

# YANGIANS AND QUANTUM LOOP ALGEBRAS

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ABSTRACT. Let  $\mathfrak{g}$  be a complex, semisimple Lie algebra. Drinfeld showed that the quantum loop algebra  $U_h(L\mathfrak{g})$  of  $\mathfrak{g}$  degenerates to the Yangian  $Y_h(\mathfrak{g})$ . We strengthen this result by constructing an explicit algebra homomorphism  $\Phi$  from  $U_h(L\mathfrak{g})$  to the completion of  $Y_h(\mathfrak{g})$  with respect to its grading. We show moreover that  $\Phi$  becomes an isomorphism when  $U_h(L\mathfrak{g})$  is completed with respect to its evaluation ideal. We construct a similar homomorphism for  $\mathfrak{g} = \mathfrak{gl}_n$  and show that it intertwines the geometric actions of the corresponding quantum loop algebra and Yangian on the equivariant  $K$ -theory and homology of the variety of  $n$ -step flags in  $\mathbb{C}^d$ .

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## 1. INTRODUCTION

1.1. The present paper is motivated by, and lays the groundwork for a proof of the trigonometric monodromy conjecture formulated by the second author in [21]. Let  $\mathfrak{g}$  be a complex, semisimple Lie algebra,  $G$  the corresponding connected and simply-connected Lie group,  $H \subset G$  a maximal torus and  $W$  the corresponding Weyl group. In [21] a flat,  $W$ -equivariant connection  $\widehat{\nabla}_C$  was constructed on  $H$  which has logarithmic singularities on the root subtori of  $H$  and values in any finite-dimensional representation of the Yangian  $Y_h(\mathfrak{g})$ . By analogy with the description of the monodromy of the Casimir rational connection of  $\mathfrak{g}$  obtained in [19, 20], it was conjectured in [21] that the monodromy of the trigonometric Casimir connection  $\widehat{\nabla}_C$  is described by the action of the affine braid group of  $\mathfrak{g}$  arising from the quantum

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Both authors are supported by NSF grants DMS-0707212 and DMS-0854792.

Weyl group operators of the quantum loop algebra  $U_h(L\mathfrak{g})$ . This naturally raises the problem of relating finite-dimensional representations of  $Y_h(\mathfrak{g})$  and  $U_h(L\mathfrak{g})$ .

1.2. Since their construction by V. Drinfeld [5, 6], these quantum groups have been extensively studied from several perspectives (see, *e.g.*, [3, chap. 12] and references therein) and are widely believed to share the same finite-dimensional representation theory. This belief is corroborated in part by the following facts

- (1) The quantum loop algebra  $U_h(L\mathfrak{g})$  degenerates to the Yangian  $Y_h(\mathfrak{g})$  [6, 11].
- (2) Finite-dimensional simple modules over  $U_h(L\mathfrak{g})$  are parametrised by  $\mathbf{I}$ -tuples of polynomials  $\{P_i(u)\}_{i \in \mathbf{I}}$  satisfying  $P_i(0) = 1$ , where  $\mathbf{I}$  is the set of vertices of the Dynkin diagram of  $\mathfrak{g}$  [2]. Similarly, finite-dimensional simple modules over  $Y_h(\mathfrak{g})$  are classified by  $\mathbf{I}$ -tuples of monic polynomials [7, 1].
- (3) If  $\mathfrak{g}$  is simply-laced, there exists, for every  $\mathbf{w} \in \mathbb{N}^{\mathbf{I}}$ , a Steinberg type variety  $Z(\mathbf{w})$  endowed with an action of  $GL(\mathbf{w}) \times \mathbb{C}^\times$  (here  $GL(\mathbf{w}) = \prod_{i \in \mathbf{I}} GL_{w_i}$ ) and algebra homomorphisms

$$\Psi_U : U_h(L\mathfrak{g}) \rightarrow K^{GL(\mathbf{w}) \times \mathbb{C}^\times}(Z(\mathbf{w}))$$

$$\Psi_Y : Y_h(\mathfrak{g}) \rightarrow H^{GL(\mathbf{w}) \times \mathbb{C}^\times}(Z(\mathbf{w}))$$

The variety  $Z(\mathbf{w})$  and the homomorphism  $\Psi_U$  were constructed by H. Nakajima [17] while  $\Psi_Y$  was constructed by M. Varagnolo [22].

1.3. The above results go some way towards relating the categories of finite-dimensional representations of  $U_h(L\mathfrak{g})$  and  $Y_h(\mathfrak{g})$ . For example, exponentiating the roots of Drinfeld polynomials yields, via (2), a surjective map  $\exp^*$  between the set of isomorphism classes of finite-dimensional modules of  $Y_h(\mathfrak{g})$  and those of  $U_h(L\mathfrak{g})$ -modules. Moreover, as pointed out in [22], the geometric realisations imply that  $\exp^*$  preserves the characters of these finite-dimensional representations under the action of the torus  $H$ .

Despite the above results however, and to the best of our knowledge, no natural relation between the categories of finite-dimensional representations of  $U_h(L\mathfrak{g})$  and  $Y_h(\mathfrak{g})$  is known. Part of the difficulty in exploiting the geometric realisations (3) to pursue this question lies in the fact that, as pointed out in [17], the homomorphisms  $\Psi_U, \Psi_Y$  are neither injective nor surjective. Moreover, although these realisations yield all irreducible representations, the categories  $\text{Rep}_{\text{fd}}(U_h(L\mathfrak{g}))$  and  $\text{Rep}_{\text{fd}}(Y_h(\mathfrak{g}))$  are not semisimple.

1.4. The aim of the present paper is to construct an algebra homomorphism

$$\Phi : U_h(L\mathfrak{g}) \longrightarrow \widehat{Y_h(\mathfrak{g})}$$

where  $\widehat{Y_h(\mathfrak{g})}$  is the completion of  $Y_h(\mathfrak{g})$  with respect to its  $\mathbb{N}$ -grading. The study of the corresponding pull-back functor

$$F = \Phi^* : \text{Rep}_{\text{fd}}(Y_h(\mathfrak{g})) \rightarrow \text{Rep}_{\text{fd}}(U_h(L\mathfrak{g}))$$

will be carried out in the sequel to this paper [9].

In more detail, recall that  $U_h(L\mathfrak{g})$  and  $Y_h(\mathfrak{g})$  are deformations of the enveloping algebras  $U(\mathfrak{g}[z, z^{-1}])$  and  $U(\mathfrak{g}[s])$  respectively, and denote by

$$U_h(L\mathfrak{h}), U_h(L\mathfrak{b}_\pm) \subset U_h(L\mathfrak{g}) \quad \text{and} \quad Y_h(\mathfrak{h}), Y_h(\mathfrak{b}_\pm) \subset Y_h(\mathfrak{g})$$

the subalgebras deforming  $U(\mathfrak{h}[z, z^{-1}])$ ,  $U(\mathfrak{b}_\pm[z, z^{-1}])$  and  $U(\mathfrak{h}[s])$ ,  $U(\mathfrak{b}_\pm[s])$  respectively, where  $\mathfrak{h} \subset \mathfrak{g}$  is the Lie algebra of  $H$  and  $\mathfrak{b}_\pm \subset \mathfrak{g}$  are the opposite Borel subalgebras corresponding to a choice  $\{\alpha_i\}_{i \in \mathbf{I}}$  of simple roots of  $\mathfrak{g}$ . For any  $i \in \mathbf{I}$ , let  $\mathfrak{sl}_2^i \subset \mathfrak{g}$  be the corresponding 3-dimensional subalgebra and denote by

$$U_h(L\mathfrak{sl}_2^i) \subset U_h(L\mathfrak{g}) \quad \text{and} \quad Y_h(\mathfrak{sl}_2^i) \subset Y_h(\mathfrak{g})$$

the subalgebras which deform  $U(\mathfrak{sl}_2^i[z, z^{-1}])$  and  $U(\mathfrak{sl}_2^i[s])$  respectively. Then, the main result of this paper is the following

**Theorem.** *There exists an explicit algebra homomorphism  $\Phi : U_h(L\mathfrak{g}) \rightarrow \widehat{Y_h(\mathfrak{g})}$  with the following properties*

- (1)  $\Phi$  is defined over  $\mathbb{Q}[[\hbar]]$ .
- (2)  $\Phi$  induces an isomorphism  $\widehat{U_h(L\mathfrak{g})} \rightarrow \widehat{Y_h(\mathfrak{g})}$ , where  $\widehat{U_h(L\mathfrak{g})}$  is the completion of  $U_h(L\mathfrak{g})$  with respect to the ideal of  $z = 1$ .
- (3)  $\Phi$  induces Drinfeld's degeneration of  $U_h(L\mathfrak{g})$  to  $Y_h(\mathfrak{g})$ .
- (4)  $\Phi$  restricts to a homomorphism  $U_h(L\mathfrak{h}) \rightarrow \widehat{Y_h(\mathfrak{h})}$  which induces the exponentiation of roots on Drinfeld polynomials.
- (5)  $\Phi$  restricts to a homomorphism  $U_h(L\mathfrak{b}_\pm) \rightarrow \widehat{Y_h(\mathfrak{b}_\pm)}$ .
- (6)  $\Phi$  restricts to a homomorphism  $U_h(L\mathfrak{sl}_2^i) \rightarrow \widehat{Y_h(\mathfrak{sl}_2^i)}$  for any  $i \in \mathbf{I}$ .

1.5. The homomorphism  $\Phi$  has the following form. Let  $\{E_{i,r}, F_{i,r}, H_{i,r}\}_{i \in \mathbf{I}, r \in \mathbb{Z}}$  be the loop generators of  $U_h(L\mathfrak{g})$  and  $\{x_{i,r}^\pm, \xi_{i,r}\}_{i \in \mathbf{I}, r \in \mathbb{N}}$  those of  $Y_h(\mathfrak{g})$  (see [7] and Section 2 for definitions). Then,

$$\begin{aligned} \Phi(H_{i,r}) &= \frac{\hbar}{q_i - q_i^{-1}} \sum_{k \geq 0} t_{i,k} \frac{r^k}{k!} \\ \Phi(E_{i,r}) &= e^{r\sigma_i^+} \sum_{m \geq 0} g_{i,m}^+ x_{i,m}^+ \\ \Phi(F_{i,r}) &= e^{r\sigma_i^-} \sum_{m \geq 0} g_{i,m}^- x_{i,m}^- \end{aligned}$$

In the formulae above,  $q = e^{\hbar/2}$  and  $q_i = q^{d_i}$ , where the  $d_i$  are the symmetrising integers for the Cartan matrix of  $\mathfrak{g}$ . The  $\{t_{i,r}\}_{i \in \mathbf{I}, r \in \mathbb{N}}$  are an alternative set of generators of the commutative subalgebra  $Y_h(\mathfrak{h}) \subset Y_h(\mathfrak{g})$  generated by the elements  $\{\xi_{i,r}\}_{i \in \mathbf{I}, r \in \mathbb{N}}$ . They are defined in [14] by equating the generating functions

$$\hbar \sum_{r \geq 0} t_{i,r} u^{-r-1} = \log(1 + \hbar \sum_{r \geq 0} \xi_{i,r} u^{-r-1})$$

The elements  $\{g_{i,m}^\pm\}_{i \in \mathbf{I}, m \in \mathbb{N}}$  lie in the completion of  $Y_{\hbar}(\mathfrak{h})$ , and are constructed as follows. Consider the formal power series

$$G(v) = \log \left( \frac{v}{e^{v/2} - e^{-v/2}} \right) \in \mathbb{Q}[[v]]$$

and define  $\gamma_i(v) \in \widehat{Y^0[v]}$  by

$$\gamma_i(v) = \hbar \sum_{r \geq 0} \frac{t_{i,r}}{r!} \left( -\frac{d}{dv} \right)^{r+1} G(v)$$

Then,

$$\sum_{m \geq 0} g_{i,m}^\pm v^m = \left( \frac{\hbar}{q_i - q_i^{-1}} \right)^{1/2} \exp \left( \frac{\gamma_i(v)}{2} \right) \quad (1.1)$$

Finally,  $\sigma_i^\pm$  are the homomorphisms of the subalgebras  $Y_{\hbar}(\mathfrak{b}_\pm) \subset Y_{\hbar}(\mathfrak{g})$  generated by  $\{\xi_{j,r}, x_{j,r}^\pm\}_{j \in \mathbf{I}, r \in \mathbb{N}}$ , which fix the  $\xi_{j,r}$  and act on the remaining generators as the shifts  $x_{j,r}^\pm \rightarrow x_{j,r+\delta_{ij}}^\pm$ .

1.6. We also construct a similar homomorphism for  $\mathfrak{g} = \mathfrak{gl}_n$  by relying on the geometric realisation of  $U_{\hbar}(L\mathfrak{gl}_n)$  obtained by V. Ginzburg and E. Vasserot [10, 23]. More precisely, fix a positive integer  $d \in \mathbb{N}$  and let  $\mathcal{F}$  be the variety of  $n$ -step flags in  $\mathbb{C}^d$ ,

$$\mathcal{F} = \{0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n = \mathbb{C}^d\}$$

The cotangent bundle  $T^*\mathcal{F}$  may be realised as

$$T^*\mathcal{F} = \{(V_\bullet, x) \in \mathcal{F} \times \text{End}(\mathbb{C}^d) \mid x(V_i) \subset V_{i-1}\}$$

and therefore admits a morphism  $T^*\mathcal{F} \rightarrow \mathcal{N}$  via the second projection, where  $\mathcal{N} = \{x \in \text{End}(\mathbb{C}^d) \mid x^n = 0\}$  is the cone of  $n$ -step nilpotent endomorphisms. Define the Steinberg variety  $Z = T^*\mathcal{F} \times_{\mathcal{N}} T^*\mathcal{F}$ . The group  $GL_d \times \mathbb{C}^\times$  acts on  $T^*\mathcal{F}$  and  $Z$  and there are surjective algebra homomorphisms

$$\Psi_U : U_{\hbar}(L\mathfrak{gl}_n) \rightarrow K^{GL_d \times \mathbb{C}^\times}(Z)$$

$$\Psi_Y : Y_{\hbar}(\mathfrak{gl}_n) \rightarrow H^{GL_d \times \mathbb{C}^\times}(Z)$$

see [10, 23] for the definition of  $\Psi_U$ .

To understand these more explicitly, one can use the convolution actions of  $K^{GL_d \times \mathbb{C}^\times}(Z)$  on  $K^{GL_d \times \mathbb{C}^\times}(T^*\mathcal{F})$  and of  $H^{GL_d \times \mathbb{C}^\times}(Z)$  on  $H^{GL_d \times \mathbb{C}^\times}(T^*\mathcal{F})$ , which are faithful. By using the equivariant Chern character, we construct an algebra homomorphism

$$\Phi : U_{\hbar}(L\mathfrak{gl}_n) \rightarrow \widehat{Y_{\hbar}(\mathfrak{gl}_n)}$$

which intertwines these two actions.

1.7. In the sequel to this paper [9], we shall prove that the pull-back functor  $F = \Phi^*$  converges for numerical values of  $\hbar$  and therefore defines a functor

$$F_a = \Phi^* : \text{Rep}_{\text{fd}}(Y_a \mathfrak{g}) \rightarrow \text{Rep}_{\text{fd}}(U_{\epsilon}(L\mathfrak{g}))$$

where  $Y_a \mathfrak{g}$  is the specialisation of  $Y_{\hbar}(\mathfrak{g})$  at  $\hbar = a \in \mathbb{C} \setminus 2\pi i\mathbb{Q}$  and  $\epsilon = \exp(a/2)$  [9].

**1.8. Outline of the paper.** In Section 2, we review the definition of the quantum loop algebra  $U_h(L\mathfrak{g})$  and Yangian  $Y_h(\mathfrak{g})$  of a semisimple Lie algebra  $\mathfrak{g}$ . We also introduce shift homomorphisms of the subalgebras  $Y_h(\mathfrak{b}_\pm)$  and straightening homomorphisms of the subalgebra  $Y_h(\mathfrak{h})$ .

In Section 3, we consider assignments mapping the generators of  $U_h(L\mathfrak{g})$  to  $\widehat{Y_h(\mathfrak{g})}$ . These have the form described in 1.5, but where the elements  $g_{i,m}^\pm \in \widehat{Y_h(\mathfrak{h})}$  are not necessarily given by (1.1). Our main result, Theorem 3.3, gives necessary and sufficient conditions for these elements to give rise to an algebra homomorphism. We call such homomorphisms *of geometric type* since, for  $\mathfrak{g}$  simply-laced, they are related to the Chern character in the geometric realisation described in 1.2.

The proof that the elements given by (1.1) satisfy the conditions of Theorem 3.3, and therefore give rise to an algebra homomorphism  $\Phi$ , is given in Section 4 (Theorem 4.6). We also prove that the action of  $\Phi$  on Drinfeld polynomials exponentiates their roots (Corollary 4.3).

In Section 5, we prove the essential uniqueness of homomorphisms of geometric type by showing that any two differ by conjugation by an element of the torus  $H$  and an invertible element of  $\widehat{Y_h(\mathfrak{h})}$  (Theorem 5.11).

In Section 6, we show that any homomorphism of geometric type  $\Phi$  induces an isomorphism  $\widehat{U_h(L\mathfrak{g})} \rightarrow \widehat{Y_h(\mathfrak{g})}$ , where  $\widehat{U_h(L\mathfrak{g})}$  is the completion with respect to the ideal of  $z = 1$  (Theorem 6.2). We show moreover that the associated graded map coincides with Drinfeld's degeneration of  $U_h(L\mathfrak{g})$  to  $Y_h(\mathfrak{g})$  (Proposition 6.5).

Section 7 contains similar results for  $\mathfrak{g} = \mathfrak{gl}_n$ . In addition to constructing an explicit homomorphism  $\Phi : U_h(L\mathfrak{gl}_n) \rightarrow \widehat{Y_h(\mathfrak{gl}_n)}$  (Theorem 7.6), we review the geometric realisations of these algebras and show that  $\Phi$  intertwines them (Theorem 7.19).

Appendix A contains a proof of the Serre relations which is required to complete the proof of Theorem 3.3.

**1.9. Acknowledgments.** We are very grateful to Ian Grojnowski from whom we learned that the quantum loop algebra and Yangian should be isomorphic after appropriate completions. His explanations and friendly insistence helped us overcome our initial doubts. We are also grateful to V. Drinfeld for showing us a proof that finite-dimensional representations separate elements of the Yangian and allowing us to reproduce it in Appendix A. We would also like to thank N. Guay for sharing a preliminary version of his preprint [11] and E. Vasserot for useful discussions.

## 2. QUANTUM LOOP ALGEBRAS AND YANGIANS

**2.1.** Let  $\mathfrak{g}$  be a complex, semisimple Lie algebra and  $(\cdot, \cdot)$  a non-degenerate, invariant bilinear form on  $\mathfrak{g}$ . Let  $\mathfrak{h} \subset \mathfrak{g}$  be a Cartan subalgebra of  $\mathfrak{g}$ ,  $\{\alpha_i\}_{i \in \mathbf{I}} \subset \mathfrak{h}^*$  a basis of simple roots of  $\mathfrak{g}$  relative to  $\mathfrak{h}$  and  $a_{ij} = 2(\alpha_i, \alpha_j)/(\alpha_i, \alpha_i)$  the entries of the corresponding Cartan matrix  $\mathbf{A}$ . Set  $d_i = (\alpha_i, \alpha_i)/2$ , so that  $d_i a_{ij} = d_j a_{ji}$  for any  $i, j \in \mathbf{I}$ . Let  $\nu : \mathfrak{h} \rightarrow \mathfrak{h}^*$  be the isomorphism determined by the inner product  $(\cdot, \cdot)$  and set  $h_i = \nu^{-1}(\alpha_i)/d_i$ . Choose root vectors  $e_i \in \mathfrak{g}_{\alpha_i}, f_i \in \mathfrak{g}_{-\alpha_i}$  such that  $[e_i, f_i] = h_i$ .

Recall that  $\mathfrak{g}$  is presented on generators  $\{e_i, f_i, h_i\}$  subject to the relations

$$\begin{aligned} [h_i, h_j] &= 0 \\ [h_i, e_j] &= a_{ij}e_j \quad [h_i, f_j] = -a_{ij}f_j \\ [e_i, f_j] &= \delta_{ij}h_i \end{aligned}$$

for any  $i, j \in \mathbf{I}$  and, for any  $i \neq j \in \mathbf{I}$

$$\begin{aligned} \text{ad}(e_i)^{1-a_{ij}}e_j &= 0 \\ \text{ad}(f_i)^{1-a_{ij}}f_j &= 0 \end{aligned}$$

A closely related, but slightly less standard presentation may be obtained by setting  $t_i = \nu^{-1}(\alpha_i) = d_i h_i$  and choosing, for any  $i \in \mathbf{I}$ , root vectors  $x_i^\pm \in \mathfrak{g}_{\pm\alpha_i}$  such that  $[x_i^+, x_i^-] = t_i$ . Then  $\mathfrak{g}$  is presented on  $\{x_i^\pm, t_i\}_{i \in \mathbf{I}}$  subject to the relations

$$\begin{aligned} [t_i, t_j] &= 0 \\ [t_i, x_j^\pm] &= \pm d_i a_{ij} x_j^\pm \\ [x_i^+, x_j^-] &= \delta_{ij} t_i \\ \text{ad}(x_i^\pm)^{1-a_{ij}} x_j^\pm &= 0 \end{aligned}$$

2.2. Throughout this paper,  $q$  and  $\hbar$  are formal variables related by  $q^2 = e^\hbar$ . For any  $i \in \mathbf{I}$ , we set  $q_i = q^{d_i} = e^{\hbar d_i/2}$ . We use the standard notation for Gaussian integers

$$\begin{aligned} [n]_q &= \frac{q^n - q^{-n}}{q - q^{-1}} \\ [n]_q! &= [n]_q [n-1]_q \cdots [1]_q \quad \left[ \begin{matrix} n \\ k \end{matrix} \right]_q = \frac{[n]_q!}{[k]_q! [n-k]_q!} \end{aligned}$$

2.3. **The quantum loop algebra.** Let  $U_\hbar(L\mathfrak{g})$  be the algebra over  $\mathbb{C}[[\hbar]]$  topologically generated by elements  $\{E_{i,k}, F_{i,k}, H_{i,k}\}_{i \in \mathbf{I}, k \in \mathbb{Z}}$  subject to the following relations

(QL1) For  $i, j \in \mathbf{I}$  and  $r, s \in \mathbb{Z}$

$$[H_{i,r}, H_{j,s}] = 0$$

(QL2) For any  $i, j \in \mathbf{I}$  and  $k \in \mathbb{Z}$ ,

$$[H_{i,0}, E_{j,k}] = a_{ij} E_{j,k} \quad [H_{i,0}, F_{j,k}] = -a_{ij} F_{j,k}$$

(QL3) For any  $i, j \in \mathbf{I}$  and  $r \in \mathbb{Z}^\times$ ,

$$[H_{i,r}, E_{j,k}] = \frac{[ra_{ij}]_{q_i}}{r} E_{j,r+k} \quad [H_{i,r}, F_{j,k}] = -\frac{[ra_{ij}]_{q_i}}{r} F_{j,r+k}$$

(QL4) For  $i, j \in \mathbf{I}$  and  $k, l \in \mathbb{Z}$

$$\begin{aligned} E_{i,k+1} E_{j,l} - q_i^{a_{ij}} E_{j,l} E_{i,k+1} &= q_i^{a_{ij}} E_{i,k} E_{j,l+1} - E_{j,l+1} E_{i,k} \\ F_{i,k+1} F_{j,l} - q_i^{-a_{ij}} F_{j,l} F_{i,k+1} &= q_i^{-a_{ij}} F_{i,k} F_{j,l+1} - F_{j,l+1} F_{i,k} \end{aligned}$$

(QL5) For  $i, j \in \mathbf{I}$  and  $k, l \in \mathbb{Z}$

$$[E_{i,k}, F_{j,l}] = \delta_{ij} \frac{\psi_{i,k+l} - \phi_{i,k+l}}{q_i - q_i^{-1}}$$

(QL6) Let  $i \neq j \in \mathbf{I}$  and set  $m = 1 - a_{ij}$ . For every  $k_1, \dots, k_m \in \mathbb{Z}$  and  $l \in \mathbb{Z}$

$$\begin{aligned} \sum_{\pi \in \mathfrak{S}_m} \sum_{s=0}^m (-1)^s \begin{bmatrix} m \\ s \end{bmatrix}_{q_i} E_{i,k_{\pi(1)}} \cdots E_{i,k_{\pi(s)}} E_{j,l} E_{i,k_{\pi(s+1)}} \cdots E_{i,k_{\pi(m)}} &= 0 \\ \sum_{\pi \in \mathfrak{S}_m} \sum_{s=0}^m (-1)^s \begin{bmatrix} m \\ s \end{bmatrix}_{q_i} F_{i,k_{\pi(1)}} \cdots F_{i,k_{\pi(s)}} F_{j,l} F_{i,k_{\pi(s+1)}} \cdots F_{i,k_{\pi(m)}} &= 0 \end{aligned}$$

where the elements  $\psi_{i,r}, \phi_{i,r}$  are defined by

$$\begin{aligned} \psi_i(z) &= \sum_{r \geq 0} \psi_{i,r} z^{-r} = \exp \left( \frac{\hbar d_i}{2} H_{i,0} \right) \exp \left( (q_i - q_i^{-1}) \sum_{s \geq 1} H_{i,s} z^{-s} \right) \\ \phi_i(z) &= \sum_{r \geq 0} \phi_{i,-r} z^r = \exp \left( -\frac{\hbar d_i}{2} H_{i,0} \right) \exp \left( -(q_i - q_i^{-1}) \sum_{s \geq 1} H_{i,-s} z^s \right) \end{aligned}$$

with  $\psi_{i,-k} = \phi_{i,k} = 0$  for every  $k \geq 1$ .

We shall denote by  $U^0 \subset U_{\hbar}(L\mathfrak{g})$  the commutative subalgebra generated by the elements  $\{H_{i,r}\}_{i \in \mathbf{I}, r \in \mathbb{Z}}$ .

**2.4. The Yangian.** Let  $Y_{\hbar}(\mathfrak{g})$  be the  $\mathbb{C}[\hbar]$ -algebra generated by elements  $\{x_{i,r}^{\pm}, \xi_{i,r}\}_{i \in \mathbf{I}, r \in \mathbb{N}}$ , subject to the following relations

(Y1) For any  $i, j \in \mathbf{I}$  and  $r, s \in \mathbb{N}$

$$[\xi_{i,r}, \xi_{j,s}] = 0$$

(Y2) For  $i, j \in \mathbf{I}$  and  $s \in \mathbb{N}$

$$[\xi_{i,0}, x_{j,s}^{\pm}] = \pm d_i a_{ij} x_{j,s}^{\pm}$$

(Y3) For  $i, j \in \mathbf{I}$  and  $r, s \in \mathbb{N}$

$$[\xi_{i,r+1}, x_{j,s}^{\pm}] - [\xi_{i,r}, x_{j,s+1}^{\pm}] = \pm \frac{d_i a_{ij} \hbar}{2} (\xi_{i,r} x_{j,s}^{\pm} + x_{j,s}^{\pm} \xi_{i,r})$$

(Y4) For  $i, j \in \mathbf{I}$  and  $r, s \in \mathbb{N}$

$$[x_{i,r+1}^{\pm}, x_{j,s}^{\pm}] - [x_{i,r}^{\pm}, x_{j,s+1}^{\pm}] = \pm \frac{d_i a_{ij} \hbar}{2} (x_{i,r}^{\pm} x_{j,s}^{\pm} + x_{j,s}^{\pm} x_{i,r}^{\pm})$$

(Y5) For  $i, j \in \mathbf{I}$  and  $r, s \in \mathbb{N}$

$$[x_{i,r}^{+}, x_{j,s}^{-}] = \delta_{ij} \xi_{i,r+s}$$

(Y6) Let  $i \neq j \in \mathbf{I}$  and set  $m = 1 - a_{ij}$ . For any  $r_1, \dots, r_m \in \mathbb{N}$  and  $s \in \mathbb{N}$

$$\sum_{\pi \in \mathfrak{S}_m} \left[ x_{i,r_{\pi(1)}}^{\pm}, \left[ x_{i,r_{\pi(2)}}^{\pm}, \dots, \left[ x_{i,r_{\pi(m)}}^{\pm}, x_{j,s}^{\pm} \right] \cdots \right] \right] = 0$$

$Y_{\hbar}(\mathfrak{g})$  is an  $\mathbb{N}$ -graded algebra by  $\deg(\xi_{i,r}) = \deg(x_{i,r}^{\pm}) = r$  and  $\deg(\hbar) = 1$ .

**2.5. PBW theorem for  $Y_h(\mathfrak{g})$ .** For any positive root  $\beta$  of  $\mathfrak{g}$ , choose a sequence of simple roots  $\alpha_{i_1}, \dots, \alpha_{i_k}$  such that  $\beta = \alpha_{i_1} + \dots + \alpha_{i_k}$  and

$$[x_{i_1}^\pm, [x_{i_2}^\pm, \dots, [x_{i_{k-1}}^\pm, x_{i_k}^\pm] \dots]] \in \mathfrak{g}_{\pm\beta}$$

are non-zero vectors. For any  $r \in \mathbb{N}$ , define  $x_{\beta,r}^\pm \in Y_h(\mathfrak{g})$  by choosing a partition  $r = r_1 + \dots + r_k$  of length  $k$  and setting

$$x_{\beta,r}^\pm = [x_{i_1,r_1}^\pm, [x_{i_2,r_2}^\pm, \dots, [x_{i_{k-1},r_{k-1}}^\pm, x_{i_k,r_k}^\pm] \dots]]$$

**Theorem ([15]).** Fix a total order on the set  $\mathcal{G} = \{\xi_{i,r}, x_{\beta,r}^\pm\}_{i \in \mathbf{I}, r \in \mathbb{N}, \beta \in \Sigma_+}$ . Then, the ordered monomials in the elements of  $\mathcal{G}$  form a basis of  $Y_h(\mathfrak{g})$ .

Let  $Y^0, Y^\pm \subset Y_h(\mathfrak{g})$  be the subalgebras generated by the elements  $\{\xi_{i,r}\}_{i \in \mathbf{I}, r \in \mathbb{N}}$  (resp.  $\{x_{i,r}^\pm\}_{i \in \mathbf{I}, r \in \mathbb{N}}$ ) and  $Y^{\geq 0}, Y^{\leq 0} \subset Y_h(\mathfrak{g})$  the subalgebras generated by  $Y^0, Y^+$  and  $Y^0, Y^-$  respectively. The following is a direct consequence of Theorem 2.5.

**Corollary.**

- (1)  $Y^0$  is a polynomial algebra in the generators  $\{\xi_{i,r}\}_{i \in \mathbf{I}, r \in \mathbb{N}}$ .
- (2)  $Y^\pm$  is the algebra generated by elements  $\{x_{i,r}^\pm\}_{i \in \mathbf{I}, r \in \mathbb{N}}$  subject to the relations  $(Y_4)$  and  $(Y_6)$ .
- (3)  $Y^{\geq 0}$  (resp.  $Y^{\leq 0}$ ) is the algebra generated by elements  $\{\xi_{i,r}, x_{i,r}^\pm\}_{i \in \mathbf{I}, r \in \mathbb{N}}$  subject to the relations  $(Y_1)$ – $(Y_4)$  and  $(Y_6)$ .
- (4) Multiplication induces an isomorphism of vector spaces

$$Y^- \otimes Y^0 \otimes Y^+ \rightarrow Y_h(\mathfrak{g})$$

**2.6. The shift operators  $\sigma_i^\pm$ .** Fix  $i \in \mathbf{I}$ . By Corollary 2.5 (3), the assignment

$$x_{j,r}^\pm \rightarrow x_{j,r+\delta_{ij}}^\pm, \quad \xi_{j,r} \rightarrow \xi_{j,r}$$

extends to an algebra homomorphism  $Y^{\geq 0} \rightarrow Y^{\geq 0}$  (resp.  $Y^{\leq 0} \rightarrow Y^{\leq 0}$ ) which we shall denote by  $\sigma_i^\pm$ .

**2.7. The relations (Y2)–(Y3).** We rewrite below the defining relations (Y2)–(Y3) of  $Y_h(\mathfrak{g})$  in terms of the shift operators  $\sigma_j^\pm$  and the generating series

$$\xi_i(u) = 1 + \hbar \sum_{r \geq 0} \xi_{i,r} u^{-r-1} \in Y_h(\mathfrak{g})[[u^{-1}]] \quad (2.1)$$

**Lemma.** The relations (Y2)–(Y3) are equivalent to

$$[\xi_i(u), x_{j,s}^\pm] = \frac{\pm \hbar d_i a_{ij}}{u - \sigma_j^\pm \pm \hbar d_i a_{ij}/2} \xi_i(u) x_{j,s}^\pm$$

where the right-hand side is expanded in powers of  $u^{-1}$ .



PROOF. Set  $a = \pm \hbar d_i a_{ij}/2$ . Multiplying (Y3) by  $\hbar u^{-r-1}$  and summing over  $r \geq 0$  yields

$$u[\xi_i(u) - 1 - \hbar u^{-1} \xi_{i,0}, x_{j,s}^\pm] - [\xi_i(u) - 1, x_{j,s+1}^\pm] = a\{x_{j,s}^\pm, \xi_i(u) - 1\}$$

where  $\{x, \xi\} = x\xi + \xi x$ . Using (Y2) and  $\{x, \xi\} = [x, \xi] + 2\xi x$ , yields

$$(u - \sigma_j^\pm + a)[\xi_i(u), x_{j,s}^\pm] = 2a\xi_i(u)x_{j,s}^\pm \quad (2.2)$$

as claimed. Conversely, taking the coefficients of  $u^0$  and  $u^{-r-1}$  in (2.2) yields (Y2) and (Y3) respectively.  $\square$

**2.8. The relations (Y4) and (Y6).** We shall use the following notation

- for an operator  $T \in \text{End}(V)$ ,  $T_{(i)} \in \text{End}(V^{\otimes m})$  is defined as

$$T_{(i)} = 1^{\otimes i-1} \otimes T \otimes 1^{\otimes m-i}$$

- for an algebra  $A$ , define  $ad^{(m)} : A^{\otimes m} \rightarrow \text{End}(A)$  as

$$ad^{(m)}(a_1 \otimes \cdots \otimes a_m) = ad(a_1) \circ \cdots \circ ad(a_m)$$

**Proposition.**

- (1) The relation (Y4) for  $i \neq j$  is equivalent to the requirement that the following holds for any  $A(v_1, v_2) \in \mathbb{C}[[v_1, v_2]]$

$$A(\sigma_i^\pm, \sigma_j^\pm)(\sigma_i^\pm - \sigma_j^\pm \mp a\hbar)x_{i,0}^\pm x_{j,0}^\pm = A(\sigma_i^\pm, \sigma_j^\pm)(\sigma_i^\pm - \sigma_j^\pm \pm a\hbar)x_{j,0}^\pm x_{i,0}^\pm$$

where  $a = d_i a_{ij}/2$ .

- (2) The relation (Y4) for  $i = j$  is equivalent to the requirement that the following holds for any  $B(v_1, v_2) \in \mathbb{C}[[v_1, v_2]]$  such that  $B(v_1, v_2) = B(v_2, v_1)$

$$\mu \left( B(\sigma_{i,(1)}^\pm, \sigma_{i,(2)}^\pm)(\sigma_{i,(1)}^\pm - \sigma_{i,(2)}^\pm \mp d_i \hbar)x_{i,0}^\pm \otimes x_{i,0}^\pm \right) = 0 \quad (2.3)$$

where  $\mu : Y_{\hbar}(\mathfrak{g})^{\otimes 2} \rightarrow Y_{\hbar}(\mathfrak{g})$  is the multiplication.

- (3) The relation (Y6) for  $i \neq j$  is equivalent to the requirement that the following holds for any  $A(v_1, \dots, v_m) \in \mathbb{C}[v_1, \dots, v_m]^{\mathfrak{S}_m}$

$$ad^{(m)} \left( A(\sigma_{i,(1)}^\pm, \dots, \sigma_{i,(m)}^\pm) \left( x_{i,0}^\pm \right)^{\otimes m} \right) x_{j,l}^\pm = 0$$

where  $m = 1 - a_{ij}$ .

PROOF. (1) The relation (Y4)

$$[x_{i,r+1}^\pm, x_{j,s}^\pm] - [x_{i,r}^\pm, x_{j,s+1}^\pm] = \pm a\hbar(x_{i,r}^\pm x_{j,s}^\pm + x_{j,s}^\pm x_{i,r}^\pm)$$

may be rewritten as

$$\sigma_i^{\pm r} \sigma_j^{\pm s} \left( \sigma_i^\pm - \sigma_j^\pm \mp a\hbar \right) x_{i,0}^\pm x_{j,0}^\pm = \sigma_i^{\pm r} \sigma_j^{\pm s} \left( \sigma_i^\pm - \sigma_j^\pm \pm a\hbar \right) x_{j,0}^\pm x_{i,0}^\pm$$

- (2) If  $i = j$ ,  $a = d_i$  and the above reads

$$\begin{aligned} \mu \left( \sigma_{i,(1)}^{\pm r} \sigma_{i,(2)}^{\pm s} \left( \sigma_{i,(1)}^\pm - \sigma_{i,(2)}^\pm \mp d_i \hbar \right) x_{i,0}^\pm \otimes x_{i,0}^\pm \right) = \\ \mu \left( \sigma_{i,(1)}^{\pm s} \sigma_{i,(2)}^{\pm r} \left( \sigma_{i,(2)}^\pm - \sigma_{i,(1)}^\pm \pm d_i \hbar \right) x_{i,0}^\pm \otimes x_{i,0}^\pm \right) \end{aligned}$$

which is equivalent to

$$\mu \left( \left( \sigma_{i,(1)}^{\pm r} \sigma_{i,(2)}^{\pm s} + \sigma_{i,(1)}^{\pm s} \sigma_{i,(2)}^{\pm r} \right) \left( \sigma_{i,(1)}^{\pm} - \sigma_{i,(2)}^{\pm} \mp d_i \hbar \right) x_{i,0}^{\pm} \otimes x_{i,0}^{\pm} \right) = 0$$

(3) is just the reformulation of (Y6).  $\square$

**Corollary.** *If (2.3) holds for some  $B \in \mathbb{C}[[v_1, v_2]]$ , then  $B(v_1, v_2) = B(v_2, v_1)$ .*

PROOF. By (2) of Proposition 2.8, we may assume that  $B(v_1, v_2) = -B(v_2, v_1)$  and therefore that  $B = (v_1 - v_2)\overline{B}$  where  $\overline{B}$  is symmetric in  $v_1 \leftrightarrow v_2$ . Using the grading on  $Y_{\hbar}(\mathfrak{g})$ , we may further assume that  $\overline{B}$  is proportional to  $v_1^r v_2^s + v_1^s v_2^r$  for some  $r \geq s \in \mathbb{N}$ . An application of (Y4) yields

$$\begin{aligned} & \mu \left( (\sigma_{i,(1)}^{\pm} - \sigma_{i,(2)}^{\pm})(\sigma_{i,(1)}^{\pm r} \sigma_{i,(2)}^{\pm s} + \sigma_{i,(1)}^{\pm s} \sigma_{i,(2)}^{\pm r})(\sigma_{i,(1)}^{\pm} - \sigma_{i,(2)}^{\pm} \mp d_i \hbar) x_{i,0}^{\pm} \otimes x_{i,0}^{\pm} \right) \\ &= 2 \left( (x_{i,r+2}^{\pm} x_{i,s}^{\pm} - x_{i,r+1}^{\pm} x_{i,s+1}^{\pm} \mp d_i \hbar x_{i,r+1}^{\pm} x_{i,s}^{\pm}) - (x_{i,r+1}^{\pm} x_{i,s+1}^{\pm} - x_{i,r}^{\pm} x_{i,s+2}^{\pm} \mp d_i \hbar x_{i,r}^{\pm} x_{i,s+1}^{\pm}) \right) \end{aligned}$$

If  $r \geq s + 2$ , the above is not zero by the PBW Theorem 2.5 and  $\overline{B} = 0$ . If  $r = s + 1$ , a further application of (Y4) shows that the second of the above two parenthesized summands is zero and again  $\overline{B} = 0$  by Theorem 2.5. Finally, if  $r = s$ , (Y4) implies that the two parenthesized summands are opposites of each other and again  $\overline{B} = 0$ .  $\square$

**2.9. An alternative system of generators for  $Y^0$ .** The following system of generators of  $Y^0$  was introduced in [14]. For any  $i \in \mathbf{I}$ , define the formal power series

$$t_i(u) = \hbar \sum_{r \geq 0} t_{i,r} u^{-r-1} \in Y^0[[u^{-1}]]$$

by

$$t_i(u) = \log(\xi_i(u)) = \log \left( 1 + \hbar \sum_{r \geq 0} \xi_{i,r} u^{-r-1} \right) \quad (2.4)$$

Since (2.4) can be inverted,  $\{t_{i,r}\}_{i \in \mathbf{I}, r \in \mathbb{N}}$  is another system of generators of  $Y^0$  which are homogeneous with  $\deg(t_{i,r}) = r$ . Moreover,  $t_{i,0} = \xi_{i,0}$  and  $t_{i,r} = \xi_{i,r} \pmod{\hbar}$  for any  $r \geq 1$  since

$$t_i(u) = \hbar \sum_{r \geq 0} \xi_{i,r} u^{-r-1} \pmod{\hbar^2}$$

In order to compute the commutation relations between  $t_{i,r}$  and  $x_{j,s}^{\pm}$ , we introduce the following formal power series (inverse Borel transform of  $t_i(u)$ )

$$B_i(v) = B(t_i(u)) = \hbar \sum_{r \geq 0} t_{i,r} \frac{v^r}{r!} \in Y^0[[v]] \quad (2.5)$$

**Lemma.** *For any  $i, j \in \mathbf{I}$  we have*

$$\left[ B_i(v), x_{j,s}^{\pm} \right] = \pm \frac{q_i^{a_{ij}v} - q_i^{-a_{ij}v}}{v} e^{\sigma_j^{\pm} v} x_{j,s}^{\pm}$$

PROOF. To simplify notation set  $a = \pm \hbar d_i a_{ij}/2$ , so that  $e^a = q_i^{\pm a_{ij}}$ . By Lemma 2.7

$$\xi_i(u) x_{j,s}^{\pm} \xi_i(u)^{-1} = \frac{u - \sigma_j^{\pm} + a}{u - \sigma_j^{\pm} - a} x_{j,s}^{\pm}$$

so that

$$[t_i(u), x_{j,s}^{\pm}] = \log \left( \frac{u - \sigma_j^{\pm} + a}{u - \sigma_j^{\pm} - a} \right) x_{j,s}^{\pm}$$

Using

$$B(\log(1 - pu^{-1})) = \frac{1 - e^{pv}}{v} \quad (2.6)$$

this yields

$$\begin{aligned} [B_i(v), x_{j,s}^{\pm}] &= B \left( \log \left( 1 - (\sigma_j^{\pm} - a)u^{-1} \right) - \log \left( 1 - (\sigma_j^{\pm} + a)u^{-1} \right) \right) x_{j,s}^{\pm} \\ &= \left( \frac{1 - e^{(\sigma_j^{\pm} - a)v}}{v} - \frac{1 - e^{(\sigma_j^{\pm} + a)v}}{v} \right) x_{j,s}^{\pm} \\ &= \frac{e^{av} - e^{-av}}{v} e^{\sigma_j^{\pm} v} x_{j,s}^{\pm} \end{aligned}$$

as claimed.  $\square$

**Remark.** In order to use Lemma 2.9 to compute the commutators  $[t_{i,r}, x_{j,s}^{\pm}]$ , one needs to expand the right-hand side as power series in  $v$ . This yields following explicit expression

$$[t_{i,r}, x_{j,s}^{\pm}] = \pm d_i a_{ij} \sum_{l=0}^{\lfloor r/2 \rfloor} \binom{r}{2l} \frac{(\hbar d_i a_{ij}/2)^{2l}}{2l+1} x_{j,r+s-2l}^{\pm}$$

These commutation relations were obtained in this form in [14, Lemma 1.4].

**2.10. The operators  $\lambda_i^{\pm}(\mathbf{v})$ .** We introduce operators  $\lambda_{i;s}^{\pm} \in \text{End}(Y^0)$  which straighten monomials of the form  $x_{i,m}^{\pm} \xi$ ,  $\xi \in Y^0$ , into elements of  $Y^0 \cdot Y^{\pm}$ .

**Proposition.** *There are operators  $\{\lambda_{i;s}^{\pm}\}_{i \in \mathbf{I}, s \in \mathbb{N}}$  on  $Y^0$  such that the following holds*

- (1) *For any  $\xi \in Y^0$ , the elements  $\lambda_{i;s}^{\pm}(\xi) \in Y^0$  are uniquely determined by the requirement that, for any  $m \in \mathbb{N}$ ,*

$$x_{i,m}^{\pm} \xi = \sum_{s \geq 0} \lambda_{i;s}^{\pm}(\xi) x_{i,m+s}^{\pm} \quad (2.7)$$

- (2) *For any  $\xi, \eta \in Y^0$ ,*

$$\lambda_{i;s}^{\pm}(\xi \eta) = \sum_{k+l=s} \lambda_{i;k}^{\pm}(\xi) \lambda_{i;l}^{\pm}(\eta) \quad (2.8)$$

- (3) *The operator  $\lambda_{i;s} : Y^0 \rightarrow Y^0$  is homogeneous of degree  $-s$ .*

(4) Let  $\lambda_i^\pm(v) : Y^0 \rightarrow Y^0[v]$  be given by

$$\lambda_i^\pm(v)(\xi) = \sum_{s \geq 0} \lambda_{i;s}^\pm(\xi) v^s$$

and extend the  $\mathbb{N}$ -grading on  $Y^0$  to  $Y^0[v]$  by  $\deg(v) = 1$ . Then  $\lambda_i^\pm(v)$  is an algebra homomorphism of degree 0.

(5)  $\lambda_{i_1}^{\epsilon_1}(v_1)$  and  $\lambda_{i_2}^{\epsilon_2}(v_2)$  commute for any  $i_1, i_2 \in \mathbf{I}$  and  $\epsilon_1, \epsilon_2 \in \{\pm 1\}$ .

(6) For any  $i \in \mathbf{I}$ ,

$$\lambda_i^+(v)\lambda_i^-(v) = \text{Id} = \lambda_i^-(v)\lambda_i^+(v)$$

(7) For any  $i, j \in \mathbf{I}$ ,

$$\lambda_j^\pm(v_1)(B_i(v_2)) = B_i(v_2) \mp \frac{q_i^{a_{ij}v_2} - q_i^{-a_{ij}v_2}}{v_2} e^{v_1 v_2}$$

(8) For any  $i \in \mathbf{I}$  and  $r \in \mathbb{N}$ ,

$$\lambda_j^\pm(v)(t_{i,r}) = t_{i,r} \mp d_i a_{ij} v^r \mod \hbar$$

PROOF. (1)–(2) by Lemma 2.7, (2.7) holds when  $\xi$  is one of the generators  $\xi_{j,r}$  of  $Y^0$ . Since (2.7) holds for  $\xi\eta$  if it holds for  $\xi, \eta \in Y^0$ , with  $\lambda_{i;s}(\xi\eta)$  given by (2.8), the  $\lambda_{i;s}$  can be defined as operators on  $Y^0$ . The fact they are uniquely characterised by (2.7) and satisfy (2.8) follows from Corollary 2.5.

(3) the linear independence of the elements on the right-hand side of (2.7) implies that  $\deg(\lambda_{i;s}(\xi)) = \deg(\xi) - s$  for any homogeneous  $\xi \in Y^0$ . (4) is a rephrasing of (2) and (3). (5) and (6) follow from (7) since the elements  $\{t_{i,n}\}$  generate  $Y^0$ . (7) follows from Lemma 2.9. (8) is a direct consequence of (7).  $\square$

**Remark.** The first assertion of the proposition above can be rephrased as:

$$x_{i,m}^\pm \xi = \lambda_i^\pm(\sigma_i^\pm)(\xi) x_{i,m}^\pm$$

### 3. CONSTRUCTION OF HOMOMORPHISMS

Let  $\widehat{Y_{\hbar}\mathfrak{g}}$  be the completion of  $Y_{\hbar}(\mathfrak{g})$  with respect to its  $\mathbb{N}$ -grading. In this section, we define an assignment

$$\Phi : \{H_{i,r}, E_{i,r}, F_{i,r}\}_{i \in \mathbf{I}, r \in \mathbb{Z}} \longrightarrow \widehat{Y_{\hbar}\mathfrak{g}}$$

and find necessary and sufficient conditions for  $\Phi$  to extend to an algebra homomorphism  $U_{\hbar}(L\mathfrak{g}) \rightarrow \widehat{Y_{\hbar}\mathfrak{g}}$ .

**3.1. Definition of  $\Phi$ .** Define

$$\Phi(H_{i,0}) = d_i^{-1} t_{i,0}$$

and, for  $r \in \mathbb{Z}^\times$

$$\Phi(H_{i,r}) = \frac{B_i(r)}{q_i - q_i^{-1}} = \frac{\hbar}{q_i - q_i^{-1}} \sum_{k \geq 0} t_{i,k} \frac{r^k}{k!}$$

where  $B_i(v)$  is the formal power series (2.5). The above assignment extends to an algebra homomorphism  $U^0 \rightarrow \widehat{Y^0}$  which will be denoted by  $\Phi^0$ .

Let now  $\{g_{i,m}^\pm\}_{i \in \mathbf{I}, m \in \mathbb{N}}$  be elements of  $\widehat{Y^0}$  and define further

$$\Phi(E_{i,0}) = \sum_{m \geq 0} g_{i,m}^+ x_{i,m}^+$$

$$\Phi(F_{i,0}) = \sum_{m \geq 0} g_{i,m}^- x_{i,m}^-$$

In terms of the shift operators  $\sigma_i^\pm$ , the above is equal to

$$\Phi(E_{i,0}) = g_i^+ (\sigma_i^+) x_{i,0}^+ \quad (3.1)$$

$$\Phi(F_{i,0}) = g_i^- (\sigma_i^-) x_{i,0}^- \quad (3.2)$$

where

$$g_i^\pm(v) = \sum_{m \geq 0} g_{i,m}^\pm v^m \in \widehat{Y^0[v]}$$

with the completion of  $Y^0[v]$  taken with respect to the  $\mathbb{N}$ -grading which extends that on  $Y^0$  by  $\deg(v) = 1$ .

If  $\Phi : U_h(L\mathfrak{g}) \rightarrow \widehat{Y_h(\mathfrak{g})}$  is an algebra homomorphism of the above form, we shall say that it is of *geometric type*.

3.2. The following result shows that the requirement that  $\Phi$  extends to an algebra homomorphism determines its value on generators  $E_{i,k}, F_{i,k}$ .

**Proposition.** *The assignment  $\Phi$  is compatible with relations (QL2)–(QL3) if, and only if*

$$\Phi(E_{i,k}) = e^{k\sigma_i^+} g_i^+ (\sigma_i^+) x_{i,0}^+ \quad (3.3)$$

$$\Phi(F_{i,k}) = e^{k\sigma_i^-} g_i^- (\sigma_i^-) x_{i,0}^- \quad (3.4)$$

PROOF. We only consider the case of the  $E'$ s. Let  $i, j \in \mathbf{I}$  and  $k \in \mathbb{Z}$ . By (Y2),

$$\begin{aligned} [\Phi(H_{i,0}), \Phi(E_{j,k})] &= [d_i^{-1} \xi_{i,0}, e^{k\sigma_j^+} g_j^+ (\sigma_j^+) x_{j,0}^+] \\ &= e^{k\sigma_j^+} g_j^+ (\sigma_j^+) [d_i^{-1} \xi_{i,0}, x_{j,0}^+] \\ &= a_{ij} \Phi(E_{j,k}) \end{aligned}$$

so that  $\Phi$  is compatible with (QL2). Next, if  $r \in \mathbb{Z}^\times$ , Lemma 2.9 yields

$$\begin{aligned} [\Phi(H_{i,r}), \Phi(E_{j,k})] &= \frac{1}{q_i - q_i^{-1}} [B_i(r), e^{k\sigma_j^+} g_j^+ (\sigma_j^+) x_{j,0}^+] \\ &= \frac{q_i^{ra_{ij}} - q_i^{-ra_{ij}}}{r(q_i - q_i^{-1})} e^{r\sigma_j^+} e^{k\sigma_j^+} g_j^+ (\sigma_j^+) x_{j,0}^+ \\ &= \frac{[ra_{ij}]_{q_i}}{r} \Phi(E_{j,r+k}) \end{aligned}$$

and  $\Phi$  is compatible with (QL3).

Conversely, if  $\Phi$  is compatible with (QL3) then  $\Phi(E_{i,r}) = r/[2r]_{q_i}[H_{i,r}, E_{i,0}]$  for  $r \neq 0$  and the computation above shows that this is equal to  $e^{r\sigma_i^+} \Phi(E_{i,0})$ .  $\square$

**Remark.** It will sometimes be convenient to write the formulae (3.3)–(3.4) as

$$\Phi(E_{i,k}) = \sum_{m \geq 0} g_{i,m}^{+, (k)} x_{i,m}^+ \quad \text{and} \quad \Phi(F_{i,k}) = \sum_{m \geq 0} g_{i,m}^{-, (k)} x_{i,m}^-$$

where the elements  $g_{i,m}^{\pm, (k)} \in \widehat{Y^0}$  are defined by

$$\sum_{m \geq 0} g_{i,m}^{\pm, (k)} v^m = e^{kv} g_i^{\pm}(v)$$

**3.3. Necessary and sufficient conditions.** Let  $\lambda_i^{\pm}(v) : Y^0 \rightarrow Y^0[v]$  be the homomorphism defined in Proposition 2.10.

**Theorem.** *The assignment  $\Phi$  given in Sections 3.1–3.2 extends to an algebra homomorphism  $U_{\hbar}(L\mathfrak{g}) \rightarrow \widehat{Y_{\hbar}(\mathfrak{g})}$  if and only if the following conditions hold*

(A) *For any  $i, j \in \mathbf{I}$*

$$g_i^+(u) \lambda_i^+(u)(g_j^-(v)) = g_j^-(v) \lambda_j^-(v)(g_i^+(u))$$

(B) *For any  $i \in \mathbf{I}$  and  $k \in \mathbb{Z}$*

$$e^{ku} g_i^+(u) \lambda_i^+(u)(g_i^-(u)) \Big|_{u^m = \xi_{i,m}} = \Phi^0 \left( \frac{\psi_{i,k} - \phi_{i,k}}{q_i - q_i^{-1}} \right)$$

(C) *For any  $i, j \in \mathbf{I}$  and  $a = d_i a_{ij}/2$*

$$g_i^{\pm}(u) \lambda_i^{\pm}(u)(g_j^{\pm}(v)) \left( \frac{e^u - e^{v \pm a\hbar}}{u - v \mp a\hbar} \right) = g_j^{\pm}(v) \lambda_j^{\pm}(v)(g_i^{\pm}(u)) \left( \frac{e^v - e^{u \pm a\hbar}}{v - u \mp a\hbar} \right)$$

PROOF. By construction and Proposition 3.2,  $\Phi$  is compatible with the relations (QL1)–(QL3). The result then follows from Lemmas 3.4 and 3.5 below and the proof of the  $q$ -Serre relations (Proposition A.1 in the Appendix).  $\square$

3.4.

**Lemma.**  *$\Phi$  is compatible with the relation (QL5) if, and only if (A) and (B) hold.*

PROOF. Compatibility with (QL5) reads

$$[\Phi(E_{i,k}), \Phi(F_{j,l})] = \delta_{ij} \Phi^0 \left( \frac{\psi_{i,k+l} - \phi_{i,k+l}}{q_i - q_i^{-1}} \right)$$

for  $i, j \in \mathbf{I}$  and  $k, l \in \mathbb{Z}$ . We begin by computing the left-hand side. By Remark 3.2,

$$\begin{aligned} \Phi(E_{i,k}) \Phi(F_{j,l}) &= \sum_{m,n \geq 0} g_{i,m}^{+, (k)} x_{i,m}^+ g_{j,n}^{-, (l)} x_{j,n}^- \\ &= \sum_{m,n,s \geq 0} g_{i,m}^{+, (k)} \lambda_{i,s}^+ \left( g_{j,n}^{-, (l)} \right) x_{i,m+s}^+ x_{j,n}^- \end{aligned}$$

and similarly

$$\Phi(F_{j,l})\Phi(E_{i,k}) = \sum_{m,n,s \geq 0} g_{j,m}^{-,(l)} \lambda_{j,s}^{-} \left( g_{i,n}^{+,(k)} \right) x_{j,m+s}^{-} x_{i,n}^{+}$$

Define  $R^{(k,l)}, L^{(k,l)} \in \widehat{Y^0}[[u, v]]$  by

$$\begin{aligned} R^{(k,l)} &= e^{ku} e^{lv} g_i^{+}(u) \lambda_i^{+}(u) (g_j^{-}(v)) \\ &= e^{ku} g_i^{+}(u) \lambda_i^{+}(u) \left( e^{lv} g_j^{-}(v) \right) \\ &= \sum_{m \geq 0} g_{i,m}^{+,(k)} u^m \sum_{s \geq 0} \lambda_{i,s}^{+} \left( \sum_{n \geq 0} g_{j,n}^{-,(l)} v^n \right) u^s \end{aligned}$$

and

$$L^{(k,l)} = e^{ku} e^{lv} g_j^{-}(v) \lambda_j^{-}(v) (g_i^{+}(u)) = \sum_{m \geq 0} g_{j,m}^{-,(l)} v^m \sum_{s \geq 0} \lambda_{j,s}^{-} \left( \sum_{n \geq 0} g_{i,n}^{+,(k)} u^n \right) v^s$$

By relation (Y5) and the PBW Theorem 2.5,  $\Phi$  is compatible with (QL5) if, and only if  $R^{(k,l)} = L^{(k,l)}$  and, for  $i = j$ ,

$$R^{(k,l)} \Big|_{u^m v^n = \xi_{i,m+n}} = \Phi^0 \left( \frac{\psi_{i,k+l} - \phi_{i,k+l}}{q_i - q_i^{-1}} \right)$$

The first equation is clearly equivalent to (A) and the second to (B).  $\square$

3.5.

**Lemma.**  $\Phi$  is compatible with the relation (QL4) if, and only if (C) holds.

PROOF. We prove the claim the  $E$ 's only. Compatibility with (QL4) reads

$$\Phi(E_{i,k+1})\Phi(E_{j,l}) - q_i^{a_{ij}} \Phi(E_{i,k})\Phi(E_{j,l+1}) = q_i^{a_{ij}} \Phi(E_{j,l})\Phi(E_{i,k+1}) - \Phi(E_{j,l+1})\Phi(E_{i,k})$$

for any  $i, j \in \mathbf{I}$  and  $k, l \in \mathbb{Z}$ . Assume first  $i \neq j$  and set  $a = d_i a_{ij}/2$  so that  $q_i^{a_{ij}} = e^{a\hbar}$ . Since

$$\Phi(E_{i,r})\Phi(E_{j,s}) = e^{r\sigma_i^{+}} e^{s\sigma_j^{+}} g_i^{+}(\sigma_i^{+}) \lambda_i^{+}(\sigma_i^{+}) \left( g_j^{+}(\sigma_j^{+}) \right) x_{i,0}^{+} x_{j,0}^{+}$$

the above reduces to

$$\begin{aligned} e^{k\sigma_i^{+}} e^{l\sigma_j^{+}} g_i^{+}(\sigma_i^{+}) \lambda_i^{+}(\sigma_i^{+}) g_j^{+}(\sigma_j^{+}) \left( e^{\sigma_i^{+}} - e^{\sigma_j^{+} + a\hbar} \right) x_{i,0}^{+} x_{j,0}^{+} \\ = e^{k\sigma_i^{+}} e^{l\sigma_j^{+}} g_j^{+}(\sigma_j^{+}) \lambda_j^{+}(\sigma_j^{+}) g_i^{+}(\sigma_i^{+}) \left( e^{\sigma_i^{+} + a\hbar} - e^{\sigma_j^{+}} \right) x_{j,0}^{+} x_{i,0}^{+} \end{aligned}$$

Using (1) of Proposition 2.8, we get

$$\begin{aligned} \left( e^{\sigma_i^{+}} - e^{\sigma_j^{+} + a\hbar} \right) x_{i,0}^{+} x_{j,0}^{+} &= \frac{e^{\sigma_i^{+}} - e^{\sigma_j^{+} + a\hbar}}{\sigma_i^{+} - \sigma_j^{+} - a\hbar} \left( \sigma_i^{+} - \sigma_j^{+} - a\hbar \right) x_{i,0}^{+} x_{j,0}^{+} \\ &= \frac{e^{\sigma_i^{+}} - e^{\sigma_j^{+} + a\hbar}}{\sigma_i^{+} - \sigma_j^{+} - a\hbar} \left( \sigma_i^{+} - \sigma_j^{+} + a\hbar \right) x_{j,0}^{+} x_{i,0}^{+} \end{aligned}$$

The PBW Theorem 2.5 then shows that the above is equivalent to (C).

Assume now that  $i = j$ , then

$$\begin{aligned}\Phi(E_{i,r})\Phi(E_{i,s}) &= \left(e^{r\sigma_i^+} g_i^+(\sigma_i^+) x_{i,0}^+\right) \left(e^{s\sigma_i^+} g_i^+(\sigma_i^+) x_{i,0}^+\right) \\ &= \mu \left(e^{r\sigma_{i,(1)}^+} e^{s\sigma_{i,(2)}^+} g_i^+(\sigma_{i,(1)}^+) \lambda_i^+(\sigma_{i,(1)}^+) \left(g_i^+(\sigma_{i,(2)}^+)\right) x_{i,0}^+ \otimes x_{i,0}^+\right)\end{aligned}$$

The compatibility with (QL4) therefore reduces to

$$\begin{aligned}\mu \left(e^{k\sigma_{i,(1)}^+} e^{l\sigma_{i,(2)}^+} g_i^+(\sigma_{i,(1)}^+) \lambda_i^+(\sigma_{i,(1)}^+) (g_i^+(\sigma_{i,(2)}^+)) \left(e^{\sigma_{i,(1)}^+} - e^{\sigma_{i,(2)}^+ + d_i \hbar}\right) x_{i,0}^+ \otimes x_{i,0}^+\right) \\ = \mu \left(e^{l\sigma_{i,(1)}^+} e^{k\sigma_{i,(2)}^+} g_i^+(\sigma_{i,(1)}^+) \lambda_i^+(\sigma_{i,(1)}^+) (g_i^+(\sigma_{i,(2)}^+)) \left(e^{\sigma_{i,(2)}^+ + d_i \hbar} - e^{\sigma_{i,(1)}^+}\right) x_{i,0}^+ \otimes x_{i,0}^+\right)\end{aligned}$$

that is to

$$\begin{aligned}\mu \left( \left( e^{k\sigma_{i,(1)}^+} e^{l\sigma_{i,(2)}^+} + e^{l\sigma_{i,(1)}^+} e^{k\sigma_{i,(2)}^+} \right) \right. \\ \left. g_i^+(\sigma_{i,(1)}^+) \lambda_i^+(\sigma_{i,(1)}^+) (g_i^+(\sigma_{i,(2)}^+)) \left( e^{\sigma_{i,(1)}^+} - e^{\sigma_{i,(2)}^+ + d_i \hbar} \right) x_{i,0}^+ \otimes x_{i,0}^+ \right) = 0\end{aligned}$$

By (2) of Proposition 2.8 and Corollary 2.8, this equation is equivalent to the requirement that

$$g_i^+(u) \lambda_i^+(u) (g_i^+(v)) \left( \frac{e^u - e^{v+d_i \hbar}}{u - v - d_i \hbar} \right)$$

be symmetric under  $u \leftrightarrow v$ , which is precisely condition (C) for  $i = j$ .  $\square$

3.6. For later use, we shall need the following

**Lemma.** Let  $\{g_i^\pm(u)\}_{i \in \mathbf{I}} \subset \widehat{Y^0[u]}$  be elements satisfying condition (B) of Theorem 3.3. Then,

$$g_i^\pm(u) = \frac{1}{d_i^\pm} \mod \widehat{Y^0[u]}_+$$

where  $\{d_i^\pm\}_{i \in \mathbf{I}} \subset \mathbb{C}^\times$  satisfy  $d_i^+ d_i^- = d_i$  for each  $i \in \mathbf{I}$ . In particular, each  $g_i^\pm(u)$  is invertible.

PROOF. Condition (B) for  $k = 0$  yields

$$g_i^+(u) \lambda_i^+(u) (g_i^-(u)) \big|_{u^m = \xi_{i,m}} = \Phi^0 \left( \frac{e^{\frac{\hbar d_i}{2} H_{i,0}} - e^{-\frac{\hbar d_i}{2} H_{i,0}}}{q_i - q_i^{-1}} \right)$$

Computing mod  $\hbar$ , and *a fortiori* mod  $\widehat{Y^0[u]}_+$ , yields

$$\Phi^0 \left( \frac{e^{\frac{\hbar d_i}{2} H_{i,0}} - e^{-\frac{\hbar d_i}{2} H_{i,0}}}{q_i - q_i^{-1}} \right) = \Phi^0(H_{i,0}) = d_i^{-1} t_{i,0}$$

Write  $g_i^\pm(u) = p_i^\pm \mod \widehat{Y^0[u]}_+$ , where  $p_i^\pm \in \mathbb{C}[t_{j,0}]_{j \in \mathbf{I}}$ . Computing mod  $\widehat{Y^0[u]}_+$ , we get

$$g_i^+(u) \lambda_i^+(u) (g_i^-(u)) \big|_{u^m = \xi_{i,m}} = p_i^+(t_{j,0}) \lambda_{i,0}^+(p_i^-(t_{j,0})) \xi_{i,0} = p_i^+(t_{j,0}) p_i^-(t_{j,0} - d_i a_{ij}) \xi_{i,0}$$



where we used (8) of Proposition 2.10. Comparing both sides and using  $\xi_{i,0} = t_{i,0}$  yields the claim.  $\square$

#### 4. EXISTENCE OF A SOLUTION

In this section, we construct an explicit homomorphism  $U_h(L\mathfrak{g}) \rightarrow \widehat{Y_h(\mathfrak{g})}$  by relying on Theorem 3.3 and showing that conditions (A)–(C) have a joint solution.

Until 4.6, we assume that  $\mathfrak{g} = \mathfrak{sl}_2$  so that  $|\mathbf{I}| = 1$ ,  $a_{ii} = 2$  and  $d_i = 1$ . To simplify notations, we drop the subscript  $i \in \mathbf{I}$ . In particular, the generators of  $U^0$  will be denoted by  $\{H_r\}_{r \in \mathbb{Z}}$  and  $\{\psi_r, \phi_{-r}\}_{r \in \mathbb{N}}$  and those of  $Y^0$  by  $\{t_r\}_{r \in \mathbb{N}}$  and  $\{\xi_r\}_{r \in \mathbb{N}}$ .

**4.1. Universal Drinfeld polynomials.** Fix an integer  $m \geq 1$ . Following [17] and [22], consider the rings

$$S(m) = \mathbb{C}[q^{\pm 1}, A_1^{\pm 1}, \dots, A_m^{\pm 1}]^{\mathfrak{S}_m}$$

$$R(m) = \mathbb{C}[\hbar, a_1, \dots, a_m]^{\mathfrak{S}_m}$$

Define a homomorphism  $\mathcal{D}^U : U^0 \rightarrow S(m)$  by

$$\mathcal{D}^U(\psi(z)) = \prod_{i=1}^m \frac{qz - q^{-1}A_i}{z - A_i} = \mathcal{D}^U(\phi(z)) \quad (4.1)$$

where the first (resp. second) equality is obtained by expanding the middle term in powers of  $z^{-1}$  and (resp.  $z$ ). Similarly, define a homomorphism  $\mathcal{D}^Y : Y^0 \rightarrow R(m)$  by

$$\mathcal{D}^Y(\xi(u)) = \prod_{i=1}^m \frac{u + \hbar - a_i}{u - a_i} \quad (4.2)$$

**Remark.** The homomorphism  $\mathcal{D}^U$  (resp.  $\mathcal{D}^Y$ ) gives the action of  $\psi(z), \phi(z)$  (resp.  $\xi(u)$ ) on the highest weight vector of the indecomposable simple  $U_h(L\mathfrak{sl}_2)$  (resp.  $Y_h(\mathfrak{sl}_2)$ ) module with Drinfeld polynomial  $(1 - A_1z) \cdots (1 - A_mz)$  (resp.  $(u - a_1) \cdots (u - a_m)$ ).

**4.2.** The following result spells out the image of the generators of  $U^0$  and  $Y^0$  under  $\mathcal{D}^U$  and  $\mathcal{D}^Y$  respectively.

**Proposition.**

(1) *The homomorphism  $\mathcal{D}^U$  maps  $H_0$  to  $m$  and, for any  $r \in \mathbb{N}^*$ ,*

$$\mathcal{D}^U(\psi_r) = (q - q^{-1}) \sum_{i=1}^m A_i^r \prod_{j \neq i} \frac{qA_i - q^{-1}A_j}{A_i - A_j} \quad (4.3)$$

$$\mathcal{D}^U(\phi_{-r}) = -(q - q^{-1}) \sum_{i=1}^m A_i^{-r} \prod_{j \neq i} \frac{qA_i - q^{-1}A_j}{A_i - A_j} \quad (4.4)$$

Moreover, for any  $r \in \mathbb{Z}^*$ ,

$$\mathcal{D}^U((q - q^{-1})H_r) = \frac{1 - q^{-2r}}{r} \sum_{i=1}^m A_i^r \quad (4.5)$$

(2) The homomorphism  $\mathcal{D}^Y$  maps  $\xi_0$  to  $m$  and, for any  $r \in \mathbb{N}^*$

$$\mathcal{D}^Y(\xi_r) = \sum_{i=1}^m a_i^r \prod_{j \neq i} \frac{a_i - a_j + \hbar}{a_i - a_j} \quad (4.6)$$

Moreover, for any  $r \in \mathbb{N}$ ,

$$\mathcal{D}^Y(t_r) = \frac{1}{r+1} \sum_{i=1}^m \frac{a_i^{r+1} - (a_i - \hbar)^{r+1}}{\hbar} \quad (4.7)$$

(3) If  $B(v) \in Y^0[[v]]$  is the series defined by (2.5), then

$$\mathcal{D}^Y(B(v)) = \frac{1 - e^{-\hbar v}}{v} \sum_{i=1}^m e^{a_i v} \quad (4.8)$$

PROOF. (1) The fact that  $\mathcal{D}^U(H_0) = m$  follows by evaluating the middle term in (4.1) at  $z = 0$ . The partial fraction decomposition of this term is readily seen to be

$$\prod_{i=1}^m \frac{qz - q^{-1}A_i}{z - A_i} = q^m + (q - q^{-1}) \sum_{i=1}^m A_i \left( \prod_{j \neq i} \frac{qA_i - q^{-1}A_j}{A_i - A_j} \right) \frac{1}{z - A_i}$$

The relations (4.3)–(4.4) follow by expanding this into positive and negative powers of  $z$  respectively. Since  $\mathcal{D}^U(\psi_0) = q^m$ , we get

$$\mathcal{D}^U \left( \exp \left( (q - q^{-1}) \sum_{s \geq 1} H_s z^{-s} \right) \right) = \mathcal{D}^U(\psi_0^{-1} \psi(z)) = \prod_{i=1}^m \frac{z - q^{-2}A_i}{z - A_i}$$

taking the log of both sides and expanding in powers of  $z^{-1}$  yields (4.5) for  $r > 0$ . The case  $r < 0$  follows by expanding in powers of  $z$ .

(2) The fact that  $\mathcal{D}^Y(\xi_0) = n$  follows by taking the coefficient of  $u^{-1}$  in (4.2). The partial fraction decomposition of  $\mathcal{D}^Y(\xi(u))$  is

$$\prod_{i=1}^m \frac{u + \hbar - a_i}{u - a_i} = 1 + \hbar \sum_{i=1}^m \left( \prod_{j \neq i} \frac{a_i - a_j + \hbar}{a_i - a_j} \right) \frac{1}{u - a_i}$$

and (4.6) follows by taking the coefficient of  $u^{-r-1}$ . Taking the log of both sides of (4.2) yields

$$\mathcal{D}^Y(t(u)) = \sum_i -\log(1 - a_i u^{-1}) + \log(1 - (a_i - \hbar)u^{-1}) \quad (4.9)$$

and therefore (4.7).

(3) The equation (4.8) follows by applying (2.6) to (4.2).  $\square$

**Corollary.** The homomorphism  $Y^0 \rightarrow \bigoplus_{m \geq 1} R(m)$  is injective.

PROOF. This follows from (4.7) and the fact that the power sums  $p_r = \sum_i a_i^r$  are algebraically independent.  $\square$

4.3. Let  $\widehat{R(m)}$  be the completion of  $R(m)$  with respect to the  $\mathbb{N}$ -grading defined by  $\deg(\hbar) = \deg(a_i) = 1$ . Since the map  $\mathcal{D}^Y : Y^0 \rightarrow R(m)$  preserves the grading, it extends to a homomorphism  $\widehat{Y^0} \rightarrow \widehat{R(m)}$ .

**Corollary.** *Let  $\text{ch} : S(m) \rightarrow R(m)$  be the algebra homomorphism defined by*

$$q \mapsto e^{\hbar/2} \quad \text{and} \quad A_i \mapsto e^{a_i}$$

*Then the following diagram commutes*

$$\begin{array}{ccc} U^0 & \xrightarrow{\mathcal{D}^U} & S(m) \\ \Phi^0 \downarrow & & \downarrow \text{ch} \\ \widehat{Y^0} & \xrightarrow{\mathcal{D}^Y} & \widehat{R(m)} \end{array}$$

where  $\Phi^0$  is defined in 3.1.

PROOF. It suffices to check the commutativity on the generators  $\{H_r\}_{r \in \mathbb{Z}}$  of  $U^0$ . The statement now follows from (4.5), (4.8) and the fact that, for  $r \neq 0$ ,  $\Phi^0(H_r) = B(v)/(q - q^{-1})|_{v=r}$ .  $\square$

4.4. **The functions  $\mathbf{G}(\mathbf{u})$  and  $\gamma(\mathbf{u})$ .** Consider the formal power series

$$G(v) = \log \left( \frac{v}{e^{v/2} - e^{-v/2}} \right) \in \mathbb{Q}[[v]] \quad (4.10)$$

Define  $\gamma(v)$  as

$$\gamma(v) = \hbar \sum_{r \geq 0} (-1)^{r+1} \frac{t_r}{r!} \partial_v^{r+1} G(v) \in \widehat{Y^0[v]}_+$$

Recall that  $B(v) = \hbar \sum_{r \geq 0} t_r \frac{v^r}{r!}$  is the inverse Borel transform of  $t(u)$ . This allows us to write  $\gamma(u)$  more compactly as

$$\gamma(v) = -B(-\partial_v)G'(v) \quad (4.11)$$

4.5.

**Proposition.** *The following holds in  $\widehat{Y^0}$  for any  $k \in \mathbb{Z}$*

$$\Phi^0 \left( \frac{\psi_k - \phi_k}{q - q^{-1}} \right) = \frac{\hbar}{q - q^{-1}} e^{kv} \exp(\gamma(v)) \Big|_{v^n = \xi_n}$$

PROOF. By Corollaries 4.2 and 4.3, it is sufficient to prove that, for any  $m \in \mathbb{N}$ ,

$$\text{ch} \left( \mathcal{D}^U \left( \frac{\psi_k - \phi_k}{q - q^{-1}} \right) \right) = \frac{\hbar}{q - q^{-1}} \mathcal{D}^Y \left( e^{kv} \exp(\gamma(v)) \Big|_{v^n = \xi_n} \right)$$

We start by computing the right-hand side. By (4.11) and (4.8),

$$\mathcal{D}^Y(\gamma(v)) = \left( \sum_i e^{-a_i \partial_v} \right) \frac{1 - e^{\hbar \partial_v}}{\partial_v} \partial_v G(v) = \sum_i G(v - a_i) - G(v - a_i + \hbar)$$

where we used  $e^{p \partial_v} G(v) = G(v + p)$ , so that

$$\mathcal{D}^Y \left( e^{kv} \exp(\gamma(v)) \right) = e^{kv} \prod_i \frac{v - a_i}{v - a_i + \hbar} \frac{q e^v - q^{-1} e^{a_i}}{e^v - e^{a_i}}$$

By Proposition 4.2, the substitution  $v^n = \xi_n$ , under  $\mathcal{D}^Y$  gives

$$\mathcal{D}^Y : F(v) \Big|_{v^n = \xi_n} \mapsto \sum_i F(a_i) \prod_{j \neq i} \frac{a_i - a_j + \hbar}{a_i - a_j}$$

which in our case implies that

$$\begin{aligned} \mathcal{D}^Y \left( e^{kv} \exp(\gamma(v)) \Big|_{v^n = \xi_n} \right) &= \frac{q - q^{-1}}{\hbar} \sum_i e^{ka_i} \prod_{j \neq i} \frac{a_i - a_j}{a_i - a_j + \hbar} \frac{q e^{a_i} - q^{-1} e^{a_j}}{e^{a_i} - e^{a_j}} \frac{a_i - a_j + \hbar}{a_i - a_j} \\ &= \frac{q - q^{-1}}{\hbar} \text{ch} \left( \sum_i A_i^k \prod_{j \neq i} \frac{q A_i - q^{-1} A_j}{A_i - A_j} \right) \end{aligned}$$

and we are done by (4.3)–(4.4) if  $k \neq 0$ . The lemma below and the fact that  $\mathcal{D}^U(\psi_0) = q^m$  settles the case  $k = 0$ .  $\square$

**Lemma.** *The following holds for any  $m \geq 1$ ,*

$$\sum_{i=1}^m \prod_{j \neq i} \frac{q A_i - q^{-1} A_j}{A_i - A_j} = [m]_q$$

PROOF. Let  $F(z) = \frac{1}{z} \prod_i \frac{q z - q^{-1} A_i}{z - A_i}$ . The partial fraction decomposition of  $F$  is

$$F(z) = \frac{q^{-m}}{z} + (q - q^{-1}) \sum_i \left( \prod_{j \neq i} \frac{q A_i - q^{-1} A_j}{A_i - A_j} \right) \frac{1}{z - A_i}$$

Multiplying both sides by  $z$  and letting  $z \rightarrow \infty$  we get

$$q^m = q^{-m} + (q - q^{-1}) \sum_{i=1}^m \prod_{j \neq i} \frac{q A_i - q^{-1} A_j}{A_i - A_j}$$

as claimed.  $\square$

**4.6. Existence of a solution.** We return now to the case of a semisimple Lie algebra  $\mathfrak{g}$  of arbitrary rank. Let  $G(v)$  be the formal power series defined in 4.4 and, for any  $i \in \mathbf{I}$  define  $\gamma_i(v) \in \widehat{Y^0[v]}_+$  by

$$\gamma_i(v) = -B_i(-\partial_v)G'(v) = \hbar \sum_{r \geq 0} (-1)^{r+1} \frac{t_{i,r}}{r!} \partial_v^{r+1} G(v)$$

Let  $g_i(v) \in \widehat{Y^0[v]}^\times$  be given by

$$g_i(v) = \left( \frac{\hbar}{q_i - q_i^{-1}} \right)^{\frac{1}{2}} \exp \left( \frac{\gamma_i(v)}{2} \right) \quad (4.12)$$

The following is the main result of this section.

**Theorem.** *The series  $g_i^\pm(v) = g_i(v)$  satisfy the conditions (A)–(C) of Theorem 3.3 and therefore give rise to an algebra homomorphism  $\Phi : U_\hbar(L\mathfrak{g}) \rightarrow \widehat{Y_\hbar \mathfrak{g}}$ .*

4.7. We shall need the following

**Lemma.** *Let  $i, j \in \mathbf{I}$  and set  $a = d_i a_{ij}/2$ . Then*

$$\lambda_i^\pm(u)(g_j(v)) = g_j(v) \exp \left( \pm \frac{G(v - u + a\hbar) - G(v - u - a\hbar)}{2} \right)$$

where  $G(v)$  is given by (4.10).

PROOF. By Proposition 2.10,

$$\lambda_i^\pm(u)(B_j(v)) = B_j(v) \mp \frac{e^{ahv} - e^{-ahv}}{v} e^{uv}$$

Since  $\gamma_j(v) = -B_j(-\partial_v)\partial_v G(v)$ , we get

$$\begin{aligned} \lambda_i^\pm(u)\gamma_j(v) &= \gamma_j(v) \pm \frac{e^{ah\partial_v} - e^{-ah\partial_v}}{\partial_v} e^{-u\partial_v} \partial_v G(v) \\ &= \gamma_j(v) \pm (G(v - u + a\hbar) - G(v - u - a\hbar)) \end{aligned}$$

The claim follows by exponentiating.  $\square$

**4.8. Proof of condition (A).** We need to prove that for every  $i, j \in \mathbf{I}$  we have

$$g_i(u)\lambda_i^+(u)(g_j(v)) = g_j(v)\lambda_j^-(v)(g_i(u))$$

By Lemma 4.7, this is equivalent to

$$\begin{aligned} g_i(u)g_j(v) \exp \left( \frac{G(v - u + a\hbar) - G(v - u - a\hbar)}{2} \right) \\ = g_i(u)g_j(v) \exp \left( \frac{G(u - v - a\hbar) - G(u - v + a\hbar)}{2} \right) \end{aligned}$$

The result now follows since  $G$  is an even function.

**4.9. Proof of condition (B).** Lemma 4.7 implies that

$$\begin{aligned} g_i(u)\lambda_i^+(u)(g_i(u)) &= g_i(u)^2 \exp\left(\frac{G(d_i\hbar) - G(-d_i\hbar)}{2}\right) \\ &= g_i(u)^2 \\ &= \frac{\hbar}{q_i - q_i^{-1}} \exp(\gamma_i(u)) \end{aligned}$$

where the second equality holds because  $G$  is even. We are done by Proposition 4.5.

**4.10. Proof of condition (C).** Let  $i, j \in \mathbf{I}$  and set  $a = d_i a_{ij}/2$ . We need to prove that

$$g_i(u)\lambda_i^\pm(u)(g_j(v)) \frac{e^u - e^{v \pm a\hbar}}{u - v \mp a\hbar} = g_j(v)\lambda_j^\pm(v)(g_i(u)) \frac{e^v - e^{u \pm a\hbar}}{v - u \mp a\hbar}$$

By Lemma 4.7 and the fact that  $G$  is even, we get the following equivalent assertion

$$\exp(G(v - u \pm a\hbar) - G(v - u \mp a\hbar)) \frac{e^u - e^{v \pm a\hbar}}{u - v \mp a\hbar} = \frac{e^v - e^{u \pm a\hbar}}{v - u \mp a\hbar}$$

Using the definition of  $G(x)$ , the above equation becomes the equality

$$\left(\frac{v - u \pm a\hbar}{e^{v \pm a\hbar} - e^u}\right) \left(\frac{e^v - e^{u \pm a\hbar}}{v - u \mp a\hbar}\right) \left(\frac{e^u - e^{v \pm a\hbar}}{u - v \mp a\hbar}\right) = \frac{e^v - e^{u \pm a\hbar}}{v - u \mp a\hbar}$$

## 5. UNIQUENESS OF HOMOMORPHISMS OF GEOMETRIC TYPE

The aim of this section is to prove that homomorphisms of geometric type are unique up to conjugation and scaling.

**5.1.** Let  $\mathcal{G}$  be the set of solutions  $\mathbf{g} = \{g_i^\pm(u)\}_{i \in \mathbf{I}}$  of equations (A)–(C) of Theorem

**3.3.** Given a collection  $\mathbf{r} = \{r_i^\pm(u)\}_{i \in \mathbf{I}}$  of invertible elements of  $\widehat{Y^0[u]}$ , set

$$\mathbf{r} \cdot \mathbf{g} = \{r_i^\pm(u) \cdot g_i^\pm(u)\}_{i \in \mathbf{I}}$$

**Lemma.** *Let  $\mathbf{g} \in \mathcal{G}$ . Then,  $\mathbf{r} \cdot \mathbf{g} \in \mathcal{G}$  if and only if the following holds*

(A<sub>0</sub>) *For any  $i, j \in \mathbf{I}$ ,*

$$r_i^+(u)\lambda_i^+(u)(r_j^-(v)) = r_j^-(v)\lambda_j^-(v)(r_i^+(u))$$

(B<sub>0</sub>) *For any  $i \in \mathbf{I}$ ,*

$$r_i^+(u)\lambda_i^+(u)(r_i^-(u)) = 1 = r_i^-(u)\lambda_i^-(u)(r_i^+(u))$$

(C<sub>0</sub><sup>±</sup>) *For any  $i, j \in \mathbf{I}$ ,*

$$r_i^\pm(u)\lambda_i^\pm(u)(r_j^\pm(v)) = r_j^\pm(v)\lambda_j^\pm(v)(r_i^\pm(u))$$

PROOF. Let  $\mathbf{h} = \mathbf{r} \cdot \mathbf{g}$ . The following assertions are straightforward to check

- $\mathbf{h}$  satisfies (A) if and only if  $\mathbf{r}$  satisfies  $(A_0)$ .
- $\mathbf{h}$  satisfies (B) if  $\mathbf{r}$  satisfies  $(B_0)$ .
- $\mathbf{h}$  satisfies (C) if and only if  $\mathbf{r}$  satisfies  $(C_0^\pm)$ .

There remains to prove that if  $\mathbf{h}$  lies in  $\mathcal{G}$ , then  $\mathbf{r}$  satisfies  $(B_0)$ .

We claim that  $(A_0)$  and  $(C_0^\pm)$  imply that  $c_i(u) = r_i^+(u)\lambda_i^+(u)(r_i^-(u))$  lies in  $\mathbb{C}[[\hbar, u]]$ . Assuming this, write  $c_i(u) = \sum_n c_i^{(n)} u^n$ , where  $c_i^{(n)} \in \mathbb{C}[[\hbar]]$ . Then,

$$\begin{aligned} (h_i^+(u)\lambda_i^+(u)h_i^-(u))|_{u^m=\xi_{i,m}} &= (c_i(u)g_i^+(u)\lambda_i^+(u)g_i^-(u))|_{u^m=\xi_{i,m}} \\ &= \sum_{n \geq 0} c_i^{(n)} (g_i^+(u)\lambda_i^+(u)g_i^-(u))|_{u^m=\xi_{i,m+n}} \\ &= c_i(\sigma_i^0) (g_i^+(u)\lambda_i^+(u)g_i^-(u))|_{u^m=\xi_{i,m}} \end{aligned}$$

where  $\sigma_i^0 : Y^0 \rightarrow Y^0$  is the algebra homomorphism defined by  $\sigma_i^0(\xi_{j,m}) = \xi_{j,m+\delta_{ij}}$ . Since both  $\mathbf{h}$  and  $\mathbf{g}$  satisfy (B) with  $k = 0$ , this yields

$$\Phi^0 \left( \frac{e^{\frac{\hbar d_i}{2} H_{i,0}} - e^{-\frac{\hbar d_i}{2} H_{i,0}}}{q_i - q_i^{-1}} \right) = c_i(\sigma_i^0) \Phi^0 \left( \frac{e^{\frac{\hbar d_i}{2} H_{i,0}} - e^{-\frac{\hbar d_i}{2} H_{i,0}}}{q_i - q_i^{-1}} \right)$$

An inductive argument using the  $\mathbb{C}[[\hbar]]$ -linear  $\mathbb{N}$ -grading on  $Y^0$  given by  $\deg(\xi_{j,m}) = m$  and  $\deg(\hbar) = 0$  then shows that  $c_i^{(0)} = 1$  and  $c_i^{(n)} = 0$  for any  $n \geq 1$ , so that  $\mathbf{r}$  satisfies  $(B_0)$ .

To prove our claim, set

$$c_i(u) = r_i^+(u)\lambda_i^+(u)(r_i^-(u)) = r_i^-(u)\lambda_i^-(u)(r_i^+(u))$$

so that  $r_i^-(u) = c_i(u)\lambda_i^-(u)(r_i^+(u))^{-1}$ . By  $(A_0)$ , the following holds for every  $i, j \in \mathbf{I}$

$$r_i^+(u)\lambda_i^+(u) \left( c_j(v)\lambda_j^-(v)(r_j^+(v))^{-1} \right) = c_j(v)\lambda_j^-(v)(r_j^+(v))^{-1}\lambda_j^-(v)(r_i^+(u))$$

Since  $\lambda_i^+(u)$  and  $\lambda_j^-(v)$  commute, we get

$$\begin{aligned} \lambda_i^+(u)(c_j(v)) \left( r_i^+(u)\lambda_j^-(v)(r_j^+(v)) \right) &= c_j(v)\lambda_j^-(v) \left( r_i^+(u)\lambda_i^+(u)(r_j^+(v)) \right) \\ &= c_j(v)\lambda_j^-(v) \left( r_j^+(v)\lambda_j^+(v)(r_i^+(u)) \right) \\ &= c_j(v) \left( r_i^+(u)\lambda_j^-(v)(r_j^+(v)) \right) \end{aligned}$$

where the second equality uses  $(C_0^+)$  and the third one  $\lambda_j^-(v)\lambda_j^+(v) = 1$ . We have therefore proved that

$$\lambda_i^+(u)(c_j(v)) = c_j(v) \quad \text{for every } i, j \in \mathbf{I}$$

By definition of the operators  $\lambda_i^\pm$ , this implies that the coefficients of  $c_j(v)$  lie in the center of  $Y_\hbar(\mathfrak{g})$ , which is trivial.  $\square$

5.2. The uniqueness of homomorphisms of geometric type relies on the following

**Proposition.** *Let  $\{r_i^+(u)\}_{i \in \mathbf{I}} \subset 1 + \widehat{Y^0[u]}_+$  be a collection of invertible elements satisfying condition  $(C_0^+)$  of Lemma 5.1. Then, there exists an element  $\xi \in 1 + \widehat{Y^0}_+$  such that, for any  $i \in \mathbf{I}$*

$$r_i^+(u) = \xi \cdot \lambda_i^+(u)(\xi)^{-1}$$

Moreover, if  $\zeta \in \widehat{Y^0}^\times$  is any element such that  $r_i^+(u) = \zeta \cdot \lambda_i^+(u)(\zeta)^{-1}$  then  $\zeta = c(\hbar)\xi$  for some  $c(\hbar) \in \mathbb{C}[[\hbar]]^\times$ .

The proof of Proposition 5.2 is given in §5.3–§5.9.

5.3. We begin by linearising the problem. Set

$$\bar{r}_i(u) = \log(r_i(u)) \in \widehat{Y^0[u]}_+$$

By condition  $(C_0^+)$ , the following holds for any  $i, j \in \mathbf{I}$

$$(\lambda_i^+(u) - 1)(\bar{r}_j(v)) = (\lambda_j^+(v) - 1)(\bar{r}_i(u)) \quad (5.1)$$

and we need to show that

$$\bar{r}_i(u) = (\lambda_i^+(u) - 1)\eta \quad (5.2)$$

for some  $\eta \in \widehat{Y^0}_+$ .

5.4. **Rank 1 case.** We assume first that  $|I| = 1$  and accordingly drop the subscript  $i$  from our formulae. We shall prove (5.2) by working with an adapted system of generators of  $Y^0$ .

Recall that, by Proposition 2.10,

$$(\lambda^+(u) - 1)B(v) = -\frac{e^{\hbar v} - e^{-\hbar v}}{v}e^{uv}$$

Define  $B'(v) = \sum_{k \geq 0} \frac{v^k}{k!} t'_k$  by equating the coefficients of  $v$  in

$$B'(v) = -\frac{v}{e^{\hbar v} - e^{-\hbar v}} Bt(v) = -\frac{\hbar v}{e^{\hbar v} - e^{-\hbar v}} \sum_{n \geq 0} \frac{v^n}{n!} t_n$$

The elements  $\{t'_k\}_{k \in \mathbb{N}}$  give another system of generators of  $Y^0$  which are homogeneous, with  $\deg(t'_k) = k = \deg(t_k)$  for any  $k \in \mathbb{N}$ , and satisfy

$$\lambda^+(u)(t'_k) = t'_k + u^k \quad (5.3)$$



5.5. Since the operator  $\lambda^+(u) : Y^0 \rightarrow Y^0[u]$  is homogeneous with respect to the  $\mathbb{N}$ -grading extending that on  $Y^0$  by  $\deg(u) = 1$ , it suffices to prove (5.2) when  $\bar{r}(u)$  is homogeneous of degree  $n \in \mathbb{N}$ . Moreover, since  $\lambda^+(u)$  is  $\mathbb{C}[\hbar]$ -linear and the formulae (5.3) do not involve  $\hbar$ , we may further assume that the coefficients of  $\bar{r}(u)$  lie in the  $\mathbb{C}$ -subalgebra  $\overline{Y^0} \subset Y^0$  generated by the  $\{t'_k\}$ .

An element of  $\overline{Y^0}[u]_n$  has the form

$$\bar{r}(u) = \sum_{|\mu| \leq n} a_\mu t'_\mu u^{n-|\mu|} \quad (5.4)$$

where  $a_\mu \in \mathbb{C}[t'_0]$  and, for a partition  $\mu$  of length  $l$ , we define  $t'_\mu = t'_{\mu_1} \cdots t'_{\mu_l}$ . The proof of the existence of  $\eta \in \overline{Y^0}_n$  such that  $(\lambda^+(u) - 1)(\eta) = \bar{r}(u)$  proceeds in two steps:

(1) Show that, modulo elements of the form  $(\lambda^+(u) - 1)(\eta)$ ,

$$\bar{r}(u) = \sum_{|\mu| < n} a_\mu t'_\mu u^{n-|\mu|} \quad (5.5)$$

where  $a_\mu \in \mathbb{C}$  do not depend on  $t'_0$ .

(2) Show that any  $\bar{r}(u)$  of the form (5.5) is equal to  $(\lambda^+(u) - 1)(\eta)$  for some  $\eta \in \overline{Y^0}_n$ .

5.6. **Proof of (1).** For  $\bar{r}(u) \in \overline{Y^0}_n$  of the form (5.4), choose  $b_\nu \in \mathbb{C}[t'_0]$  for every  $\nu \vdash n$  such that

$$b_\nu(t'_0 + 1) - b_\nu(t'_0) = a_\nu(t'_0)$$

Then

$$\bar{r}(u) - (\lambda^+(u) - 1) \left( \sum_{\nu \vdash n} b_\nu t'_\nu \right) = \sum_{|\mu| < n} a'_\mu t'_\mu u^{n-|\mu|}$$

for some  $a'_\mu \in \mathbb{C}[t'_0]$ , so that we may assume that  $\bar{r}(u)$  is of the form (5.4) with  $a_\mu = 0$  for any  $\mu \vdash n$ .

Write now

$$\begin{aligned} & (\lambda^+(v) - 1)\bar{r}(u) \\ &= \sum_{|\mu| < n} (a_\mu(t_0 + 1) - a_\mu(t_0)) t'_\mu u^{n-|\mu|} + \sum_{|\mu| < n} a_\mu(t_0 + 1) \left( \sum_{\nu \subsetneq \mu} c(\nu, \mu) t'_\nu v^{|\mu|-|\nu|} \right) u^{n-|\mu|} \end{aligned}$$

where  $c(\nu, \mu)$  is the number of ways of obtaining  $\nu$  by removing rows from  $\mu$ . By (5.1), the above expression is symmetric in  $u$  and  $v$ . Its value at  $u = 0$ , which is 0, must therefore equal its value at  $v = 0$ , thus leading to

$$\sum_{|\mu| < n} (a_\mu(t_0 + 1) - a_\mu(t_0)) t'_\mu u^{n-|\mu|} = 0$$

which implies that  $a_\mu \in \mathbb{C}$  for any  $\mu$ .

**5.7. Proof of (2).** Let  $\bar{r}(u)$  be of the form (5.5). For any  $0 \leq l \leq n$ , write

$$\bar{r}_l(u) = \sum_{\substack{|\mu| \leq n \\ l(\mu) = l}} a_\mu t'_\mu u^{n-|\mu|}$$

so that  $\bar{r}(u) = \sum_l \bar{r}_l(u)$ . We proceed by induction on the largest positive integer  $k$  such that  $\bar{r}_k(u) \neq 0$ . If  $k = 0$ , then  $\bar{r}(u) = cu^n = (\lambda^+(u) - 1)(ct'_n)$ .

Assume now that  $k > 0$  and let  $D(u) : \bar{Y}^0 \rightarrow \bar{Y}^0[u]$  be the differential operator  $D(u) = \sum_{m \geq 1} u^m \partial_{t'_m}$ . Since  $(\lambda^+(u) - 1)(t'_k) = u^k$  we get, for any partition  $\mu$

$$(\lambda^+(u) - 1)(t'_\mu) = D(u)(t'_\mu) + \text{terms of smaller length}$$

Thus, (5.1) implies that

$$D(u)\bar{r}_k(v) = D(v)\bar{r}_k(u)$$

This cross-derivative condition implies the existence of  $\eta \in \bar{Y}^0_n$  such that  $r_k(u) = D(u)\eta$ . This implies that  $\bar{r}(u) - (\lambda^+(u) - 1)(\eta)$  has smaller  $k$ .

This completes the proof of the existence part of Proposition 5.2 when  $\mathfrak{g}$  is of rank 1.

**5.8. Arbitrary rank.** The argument for arbitrary  $\mathfrak{g}$  rests on the following

**Lemma.** *There exist generators  $\{\varpi_{i,r}\}_{i \in \mathbf{I}, r \in \mathbb{N}}$  of  $Y^0$  which are homogeneous, with  $\deg(\varpi_{i,r}) = r$  and such that*

$$\lambda_i^\pm(u)\varpi_{j,r} = \varpi_{j,r} \pm \delta_{i,j}u^r$$

PROOF. By Proposition 2.10, the generating series  $B_j(v) = \hbar \sum_{r \geq 0} t_{j,r} v^r / r!$  satisfy

$$(\lambda_i^\pm(u) - 1)\hbar^{-1}B_j(v) = \mp Q_{ij}(v)e^{uv}$$

where  $Q_{ij}(v) = 2 \sinh(\hbar d_i a_{ij} v / 2) / \hbar v$ . Since  $Q_{ij} = d_i a_{ij} \bmod \hbar$ , the matrix  $Q = (Q_{ij})$  is invertible. Set  $B'_i(v) = -\hbar^{-1} \sum_j Q_{ij}^{-1} B_j(v)$ . Then  $(\lambda_i^\pm(u) - 1)B'_i(v) = \pm \delta_{ij} e^{uv}$  which, in terms of the expansion  $B'_i(v) = \sum \varpi_{i,r} v^r / r!$  yields the required transformation property.

Since  $\deg(v) = 1$ , the homogeneity of the  $\varpi_{i,r}$  is equivalent to  $\text{Ad}(\zeta)(B'_i(v)) = B'_i(\zeta^2 v)$  where  $\text{Ad}(\zeta)$  denotes the action of  $\zeta \in \mathbb{C}^\times$  on  $Y_\hbar(\mathfrak{g})[[v]]$  corresponding to the  $\mathbb{N}$ -grading. This in turn follows from the fact that  $\text{Ad}(\zeta)(\hbar^{-1}B_j(v)) = B_j(\zeta^2 v)$  and  $\text{Ad}(\zeta)Q(v) = Q(\zeta^2 v)$ .  $\square$

Using the generators  $\varpi_{i,r}$ , the proof of the existence part of Proposition 5.2 in higher rank follows the same argument as the one used for proving the sufficiency of cross derivative condition (here the existence of a primitive for any  $i \in \mathbf{I}$  is guaranteed by the rank 1 case).

**5.9. Uniqueness of  $\xi$ .** Let  $\zeta \in \widehat{Y^0}^\times$  be an element such that  $r_i^+(u) = \zeta \cdot \lambda_i^+(u)(\zeta)^{-1}$  for each  $i \in \mathbf{I}$ . Then

$$\begin{aligned} \lambda_i^+(u)(\zeta\xi^{-1}) &= \lambda_i^+(u)(\zeta)\lambda_i^+(u)(\xi)^{-1} \\ &= r_i^+(u)^{-1}\zeta r_i^+(u)\xi^{-1} \\ &= \zeta\xi^{-1} \end{aligned}$$

By Proposition 2.10 (6), we get that  $\lambda_i^\pm(u)(\zeta\xi^{-1}) = \zeta\xi^{-1}$  for each  $i \in \mathbf{I}$ . By definition of the operators  $\lambda_i^\pm(u)$ , this implies that the coefficients of  $\zeta\xi^{-1}$  lie in the center of  $Y_h(\mathfrak{g})$ , which is trivial. This completes the proof of the last assertion of Proposition 5.2.

**5.10. Torus action.** The adjoint action of  $\mathfrak{h}$  on  $Y_h(\mathfrak{g})$  exponentiates to one of the algebraic torus  $H = \text{Hom}_{\mathbb{Z}}(Q, \mathbb{C}^\times)$  where  $Q \subset \mathfrak{h}^*$  is the root lattice. This action preserves homomorphisms of geometric type and acts on the corresponding formal power series by  $\zeta \cdot \{g_i^\pm(u)\} = \{\zeta_i^\pm g_i^\pm(u)\}$  where  $H \ni \zeta \rightarrow \zeta_i = \zeta(\alpha_i)$  is the  $i$ th coordinate function on  $H$ .

**5.11. Uniqueness of homomorphisms of geometric type.**

**Theorem.** Let  $\Phi, \Phi' : U_h(L\mathfrak{g}) \rightarrow \widehat{Y_h(\mathfrak{g})}$  be two homomorphisms of geometric type. Then, there exists  $\zeta \in H$  and  $\xi \in \widehat{Y^0}^\times$  such that

$$\Phi' = \text{Ad}(\xi) \circ (\zeta \cdot \Phi)$$

Moreover,  $\zeta$  is unique and  $\xi$  is unique up to multiplication by  $c \in \mathbb{C}[[\hbar]]^\times$ .

PROOF. Let  $\{g_i^\pm(u)\}, \{h_i^\pm(u)\} \subset \widehat{Y^0[[u]]}$  be elements of  $\mathcal{G}$  corresponding to  $\Phi$  and  $\Phi'$  respectively. By Lemma 5.1, the elements  $r_i^\pm(u) = h_i^\pm(u) \cdot g_i^\pm(u)^{-1}$  satisfy conditions  $(A_0)-(C_0^\pm)$ . By Lemma 3.6, we may use the action of  $H$  to assume that  $g_i^\pm(u) = h_i^\pm(u) \bmod \widehat{Y^0[[u]]}_+$ . By Proposition 5.2, we may find an element  $\xi \in 1 + \widehat{Y^0}_+$  such that  $r_i^+(u) = \xi \cdot \lambda_i^+(u)(\xi^{-1})$ . It follows that for any  $i \in \mathbf{I}$

$$\begin{aligned} \Phi'(E_{i,0}) &= h_i^+(\sigma_i^+)x_{i,0}^+ \\ &= r_i^+(\sigma_i^+)g_i^+(\sigma_i^+)x_{i,0}^+ \\ &= \xi\lambda_i^+(\sigma_i^+)(\xi^{-1})g_i^+(\sigma_i^+)x_{i,0}^+ \\ &= \xi g_i^+(\sigma_i^+)x_{i,0}^+\xi^{-1} \end{aligned}$$

Moreover, for any  $r \in \mathbb{Z}$ ,

$$\Phi'(E_{i,r}) = e^{r\sigma_i^+} \Phi'(E_{i,0}) = e^{r\sigma_i^+} \xi \Phi(E_{i,0}) \xi^{-1} = \xi \Phi(E_{i,r}) \xi^{-1}$$

By  $(B_0)$ ,  $r_i^-(u) = \lambda_i^-(u)(r_i^+(u)^{-1}) = \xi\lambda_i^-(u)(\xi^{-1})$  and it follows similarly that  $\Phi'(F_{i,r}) = \xi\Phi(F_{i,r})\xi^{-1}$  for any  $i \in \mathbf{I}$  and  $r \in \mathbb{Z}$ . Since  $\Phi$  and  $\Phi'$  coincide on  $U^0$  and  $\text{Ad}(\xi)(\eta) = \eta$  for any  $\eta \in Y^0$  it follows that  $\Phi' = \text{Ad}(\xi) \circ \Phi$ . The last assertion of Proposition 5.2 implies the uniqueness of  $\xi$  up to multiplication by an element of  $\mathbb{C}[[\hbar]]^\times$ .  $\square$

## 6. ISOMORPHISMS OF GEOMETRIC TYPE

We prove in this section that any homomorphism of geometric type  $\Phi : U_h(L\mathfrak{g}) \rightarrow \widehat{Y_h(\mathfrak{g})}$  extends to an isomorphism of completed algebras and induces Drinfeld's degeneration of  $U_h(L\mathfrak{g})$  to  $Y_h(\mathfrak{g})$ .

**6.1. Classical limit.** The specialisations of the quantum loop algebra  $U_h(L\mathfrak{g})$  and Yangian  $Y_h(\mathfrak{g})$  at  $\hbar = 0$  are the enveloping algebras  $U(\mathfrak{g}[z, z^{-1}])$  and  $U(\mathfrak{g}[s])$  respectively. Specifically, if  $\{e_i, f_i, h_i\}_{i \in \mathbf{I}}$  are the generators of  $\mathfrak{g}$  given in Section 2.1, the assignments

$$e_i \otimes z^k \rightarrow E_{i,k}, \quad f_i \otimes z^k \rightarrow F_{i,k}, \quad h_i \otimes z^r \rightarrow H_{i,r}$$

and

$$e_i \otimes s^r \rightarrow \frac{1}{\sqrt{d_i}} x_{i,r}^+, \quad f_i \otimes s^r \rightarrow \frac{1}{\sqrt{d_i}} x_{i,r}^-, \quad h_i \otimes s^r \rightarrow \frac{1}{d_i} \xi_{i,r}$$

extend respectively to isomorphisms

$$U(\mathfrak{g}[z, z^{-1}]) \xrightarrow{\sim} U_h(L\mathfrak{g})/\hbar U_h(L\mathfrak{g}) \quad \text{and} \quad U(\mathfrak{g}[s]) \xrightarrow{\sim} Y_h(\mathfrak{g})/\hbar Y_h(\mathfrak{g})$$

**Proposition.** *Let  $\Phi : U_h(L\mathfrak{g}) \rightarrow \widehat{Y_h(\mathfrak{g})}$  be the homomorphism given by Theorem 4.6. Then, the specialisation of  $\Phi$  at  $\hbar = 0$  is the homomorphism*

$$\exp^* : U(\mathfrak{g}[z, z^{-1}]) \longrightarrow U(\mathfrak{g}[[s]]) \subset \widehat{U(\mathfrak{g}[s])}$$

given on  $\mathfrak{g}[z, z^{-1}]$  by  $\exp^*(X \otimes z^k) = X \otimes e^{ks}$ .

PROOF. Since  $\Phi(H_{i,0}) = d_i^{-1} t_{i,0}$  and, for  $r \geq 1$ ,

$$\Phi(H_{i,r}) = \frac{\hbar}{q_i - q_i^{-1}} \sum_{k \geq 0} t_{i,k} \frac{r^k}{k!}$$

setting  $\hbar = 0$  yields  $\Phi|_{\hbar=0}(h_i \otimes z^0) = h_i \otimes s^0$ , and

$$\Phi|_{\hbar=0}(h_i \otimes z^r) = \frac{1}{d_i} \sum_{k \geq 0} d_i h_i \otimes \frac{s^k r^k}{k!} = h_i \otimes e^{rs}$$

Further, since  $g_i^+(u) = \frac{1}{\sqrt{d_i}} \pmod{\hbar}$  by (4.12), we get

$$\Phi|_{\hbar=0}(e_i \otimes z^r) = \frac{1}{\sqrt{d_i}} e^{r\sigma_i^+} \sqrt{d_i} e_i \otimes s^0 = \sum_{k \geq 0} e_i \otimes \frac{s^k r^k}{k!} = e_i \otimes e^{rs}$$

where we used the fact that, in the classical limit, the operator  $\sigma_i^+$  corresponds to multiplication by  $s$ . Similarly,  $\Phi|_{\hbar=0}(f_i \otimes z^r) = f_i \otimes e^{rs}$ .  $\square$

6.2. Let  $\mathcal{J} \subset U_{\hbar}L\mathfrak{g}$  be the kernel of the composition

$$U_{\hbar}L\mathfrak{g} \xrightarrow{\hbar \rightarrow 0} U(L\mathfrak{g}) \xrightarrow{z \rightarrow 1} U\mathfrak{g}$$

and let

$$\widehat{U_{\hbar}(L\mathfrak{g})} = \varprojlim U_{\hbar}(L\mathfrak{g})/\mathcal{J}^n$$

be the completion of  $U_{\hbar}(L\mathfrak{g})$  with respect to the ideal  $\mathcal{J}$ .

**Theorem.** *Let  $\Phi : U_{\hbar}(L\mathfrak{g}) \rightarrow \widehat{Y_{\hbar}\mathfrak{g}}$  be a homomorphism of geometric type. Then,*

- (1)  $\Phi$  maps  $\mathcal{J}$  to the ideal  $\widehat{Y_{\hbar}(\mathfrak{g})}_+ = \prod_{n \geq 1} Y_{\hbar}(\mathfrak{g})_n$ .
- (2) The corresponding homomorphism

$$\widehat{\Phi} : \widehat{U_{\hbar}(L\mathfrak{g})} \rightarrow \widehat{Y_{\hbar}(\mathfrak{g})}$$

*is an isomorphism.*

PROOF. (1) Note first that  $\mathcal{J}$  is generated by  $\hbar U_{\hbar}(L\mathfrak{g})$  and the elements  $\{H_{i,r} - H_{i,s}, E_{i,r} - E_{i,s}, F_{i,r} - F_{i,s}\}_{i \in \mathbf{I}, r, s \in \mathbb{Z}}$  since its image in  $U(\mathfrak{g}[z, z^{-1}])$  is generated by the classes of these elements. Note next that, for  $r, s \neq 0$

$$\Phi(H_{i,r} - H_{i,s}) = \frac{\hbar}{q_i - q_i^{-1}} \sum_{k \geq 1} \frac{r^k - s^k}{k!} t_{i,k}$$

while

$$\Phi(H_{i,r} - H_{i,0}) = \frac{\hbar}{q_i - q_i^{-1}} \sum_{k \geq 1} \frac{r^k}{k!} t_{i,k} + \left( \frac{\hbar}{q_i - q_i^{-1}} - d_i^{-1} \right) t_{i,0}$$

which lies in  $\prod_{n \geq 1} Y_{\hbar}(\mathfrak{g})_n$  since  $\hbar/(q_i - q_i^{-1}) = d_i^{-1} \pmod{\hbar}$ . Finally, for  $r, s \in \mathbb{Z}$ ,

$$\Phi(E_{i,r} - E_{i,s}) = (e^{r\sigma_i^+} - e^{s\sigma_i^+})g_i^+(\sigma_i^+)e_{i,0} \in \mathcal{J}$$

and similarly  $\Phi(F_{i,r} - F_{i,s}) \in \mathcal{J}$ .

(2) By Theorem 5.11, it suffices to prove this for the explicit homomorphism given by Theorem 4.6. The result then follows Proposition 6.3 below and the fact that, by Proposition 6.1, the specialisation of  $\widehat{\Phi}$  at  $\hbar = 0$  is an isomorphism  $U(\widehat{\mathfrak{g}[z, z^{-1}]}) \rightarrow \widehat{U(\mathfrak{g}[s])}$ .  $\square$

6.3. Let  $J \subset U(\mathfrak{g}[z, z^{-1}])$  be the kernel of evaluation at  $z = 1$  and  $\widehat{U(L\mathfrak{g})}$  the completion of  $U(L\mathfrak{g})$  with respect to  $J$ .

**Proposition.**

- (1)  $\widehat{U_{\hbar}(L\mathfrak{g})}$  is a flat deformation of  $\widehat{U(\mathfrak{g}[z, z^{-1}])}$ .
- (2)  $\widehat{Y_{\hbar}(\mathfrak{g})}$  is a flat deformation of  $\widehat{U(\mathfrak{g}[s])}$  over  $\mathbb{C}[[\hbar]]$ .

PROOF. (1) Set for brevity  $\mathcal{U} = U_h(L\mathfrak{g})$  and  $U = U(\mathfrak{g}[z, z^{-1}])$ . By [13, Prop XVI.2.4], it suffices to prove that  $\widehat{\mathcal{U}}$  is a separated, complete and torsion-free  $\mathbb{C}[[\hbar]]$ -module. To show that it is separated, note that  $\hbar \in \mathcal{J}$ , so that  $\hbar^k \widehat{\mathcal{U}} \subset \lim_{n \geq k} \mathcal{J}^k / \mathcal{J}^n$  and

$$\bigcap_{k \geq 0} \hbar^k \widehat{\mathcal{U}} = \{0\}$$

To show completeness, note that

$$\widehat{\mathcal{U}} / \hbar^k \widehat{\mathcal{U}} = \lim_{\leftarrow n} (\mathcal{U} / \mathcal{J}^n) / (\hbar^k \mathcal{U} / \hbar^k \mathcal{U} \cap \mathcal{J}^n) = \lim_{\leftarrow n} \begin{cases} \mathcal{U} / \mathcal{J}^n & \text{if } n \leq k \\ \mathcal{U} / \hbar^k \mathcal{U} + \mathcal{J}^n & \text{if } n > k \end{cases}$$

from which it readily follows that the map

$$\widehat{\mathcal{U}} \longrightarrow \lim_{\leftarrow k} \mathcal{U} / \hbar^k \mathcal{U}$$

is surjective. Finally, to prove that  $\widehat{\mathcal{U}}$  is torsion-free, note that the kernel of multiplication by  $\hbar$  on  $\mathcal{U} / \mathcal{J}^n$  is  $\hbar^{-1}(\hbar \mathcal{U} \cap \mathcal{J}^n) / \mathcal{J}^n$ . We claim that  $\hbar \mathcal{U} \cap \mathcal{J}^n = \hbar \mathcal{J}^{n-1}$ . Assuming this for now yields that the kernel of  $\hbar$  on  $\widehat{\mathcal{U}}$  is  $\lim_n \mathcal{J}^{n-1} / \mathcal{J}^n = \{0\}$ . To prove the claim, use the flatness of  $\mathcal{U}$  to identify it with the  $\mathbb{C}[[\hbar]]$ -module  $U[[\hbar]]$ , so that  $\mathcal{J} = J \oplus \hbar U[[\hbar]]$ . Let  $a_1, \dots, a_n \in \mathcal{J}$  and write  $a_i = a_i^0 + \hbar \bar{a}_i$ , where  $a_i^0 \in J$  and  $\bar{a}_i \in U[[\hbar]]$ . Then

$$a_1 \cdots a_n = a_1^0 a_2^0 \cdots a_n^0 \mod \hbar \mathcal{J}^{n-1} = \cdots = a_1^0 \cdots a_n^0 \mod \hbar \mathcal{J}^{n-1}$$

from which the claim follows.

(2) Since  $\widehat{Y_h(\mathfrak{g})}$  is the completion of  $Y_h(\mathfrak{g})$  with respect to the ideal  $Y_h(\mathfrak{g})_+$  of elements of positive degree, it follows as in (1) that it is a separated and complete  $\mathbb{C}[[\hbar]]$ -module. The lack of torsion of  $Y_h(\mathfrak{g})$  implies that  $\hbar Y_h(\mathfrak{g}) \cap Y_h(\mathfrak{g})_+^n = \hbar Y_h(\mathfrak{g})_+^{n-1}$  and therefore that  $\widehat{Y_h(\mathfrak{g})}$  is torsion-free. Thus,  $\widehat{Y_h(\mathfrak{g})}$  is a flat deformation of

$$\widehat{Y_h(\mathfrak{g})} / \hbar \widehat{Y_h(\mathfrak{g})} \cong Y_h(\mathfrak{g}) / \hbar Y_h(\mathfrak{g}) \cong U(\mathfrak{g}[s])$$

as claimed.  $\square$

**6.4. Drinfeld's degeneration.** Consider the descending filtration

$$U_h(L\mathfrak{g}) = \mathcal{J}^0 \supset \mathcal{J} \supset \mathcal{J}^2 \supset \cdots \quad (6.1)$$

defined by the powers of  $\mathcal{J}$  and let  $\text{gr}_{\mathcal{J}}(U_h(L\mathfrak{g})) = \bigoplus_{n \geq 0} \mathcal{J}^n / \mathcal{J}^{n+1}$  be its associated graded.

**Theorem** ([6, 11]). *Let  $\{d_i^{\pm}\}_{i \in \mathbf{I}} \subset \mathbb{C}^{\times}$  be such that  $d_i^+ d_i^- = d_i$ . Then the following assignment extends uniquely to an isomorphism of graded algebras  $Y_h(\mathfrak{g}) \xrightarrow{\sim} \text{gr}_{\mathcal{J}}(U_h(L\mathfrak{g}))$*

$$\begin{aligned} \xi_{i,0} &\longmapsto d_i H_{i,0} \in U_h(L\mathfrak{g}) / \mathcal{J} \\ x_{i,0}^+ &\longmapsto d_i^+ E_{i,0} \in U_h(L\mathfrak{g}) / \mathcal{J}, & x_{i,0}^- &\longmapsto d_i^- F_{i,0} \in U_h(L\mathfrak{g}) / \mathcal{J} \\ x_{i,1}^+ &\longmapsto d_i^+ (E_{i,1} - E_{i,0}) \in \mathcal{J} / \mathcal{J}^2, & x_{i,1}^- &\longmapsto d_i^- (F_{i,1} - F_{i,0}) \in \mathcal{J} / \mathcal{J}^2 \end{aligned}$$

**Remark.** The fact that  $U_h(L\mathfrak{g})$  degenerates to  $Y_h(\mathfrak{g})$  was stated, without proof, in [6, §6]. The formulae above and the proof that they define an isomorphism  $Y_h(\mathfrak{g}) \cong \text{gr}_{\mathcal{J}}(U_h\mathfrak{g})$  are given in [11].

**6.5. Relation to Drinfeld's degeneration.** By Theorem 6.2, a homomorphism of geometric type  $\Phi$  induces a homomorphism

$$\text{gr}(\Phi) : \text{gr}_{\mathcal{J}}(U_h(L\mathfrak{g})) \longrightarrow Y_h(\mathfrak{g}) = \widehat{\text{gr}_{Y_h(\mathfrak{g})+}} \widehat{Y_h(\mathfrak{g})}$$

Let  $\{g_i^{\pm}(v)\} \subset \widehat{Y^0[v]}^{\times}$  be the elements defining  $\Phi$ . By Lemma 3.6,

$$g_i^{\pm}(v) = \frac{1}{d_i^{\pm}} \mod \widehat{Y^0[v]}_+ \quad (6.2)$$

for some  $d_i^{\pm} \in \mathbb{C}^{\times}$  such that  $d_i^+ d_i^- = d_i$ .

**Proposition.**  $\text{gr}(\Phi)$  is the inverse of the degeneration isomorphism  $\iota : Y_h(\mathfrak{g}) \xrightarrow{\sim} U_h(L\mathfrak{g})$  given by Theorem 6.4.

PROOF. It suffices to verify the claim on the generators  $\{\xi_{i,0}, x_{i,0}^{\pm}, x_{i,1}^{\pm}\}_{i \in \mathbf{I}}$  of  $Y_h(\mathfrak{g})$ . Now,

$$\text{gr}(\Phi) \circ \iota(\xi_{i,0}) = \text{gr}(\Phi)(d_i H_{i,0}) = \xi_{i,0}$$

Moreover,

$$\begin{aligned} \text{gr}(\Phi) \circ \iota(x_{i,0}^+) &= d_i^+ \Phi(E_{i,0}) \mod \widehat{Y_h(\mathfrak{g})}_+ \\ &= d_i^+ g_i^+(\sigma_i^+) x_{i,0}^+ \mod \widehat{Y_h(\mathfrak{g})}_+ \\ &= x_{i,0}^+ \end{aligned}$$

by (6.2). Moreover,

$$\begin{aligned} \text{gr}(\Phi) \circ \iota(x_{i,1}^+) &= d_i^+ \Phi(E_{i,1} - E_{i,0}) \mod \widehat{Y_h(\mathfrak{g})}_{\geq 2} \\ &= d_i^+ (e^{\sigma_i^+} - 1) g_i^+(\sigma_i^+) x_{i,0}^+ \mod \widehat{Y_h(\mathfrak{g})}_{\geq 2} \\ &= x_{i,1}^+ \end{aligned}$$

And similarly  $\text{gr}(\Phi) \circ \iota(x_{i,r}^-) = x_{i,r}^-$  for  $r = 0, 1$ .  $\square$

## 7. GEOMETRIC SOLUTION FOR $\mathfrak{gl}_n$

In this section, we construct a homomorphism of geometric type for  $\mathfrak{gl}_n$  and show that it intertwines the geometric realisations of the corresponding loop algebra and Yangian.

**7.1. The quantum loop algebra [4].** Throughout this section, we fix  $n \geq 2$  and mostly follow the notation of [23]. Set  $\mathbf{I} = \{1, \dots, n-1\}$  and  $\mathbf{J} = \{1, \dots, n\}$ . Then,  $U_h(L\mathfrak{gl}_n)$  is topologically generated over  $\mathbb{C}[[\hbar]]$  by elements  $\{E_{i,r}, F_{i,r}, D_{j,r}\}_{i \in \mathbf{I}, j \in \mathbf{J}, r \in \mathbb{Z}}$ . To describe the relations, introduce the formal power series

$$E_i(z) = \sum_{r \in \mathbb{Z}} E_{i,r} z^{-r} \quad F_i(z) = \sum_{r \in \mathbb{Z}} F_{i,r} z^{-r}$$

and

$$\Theta_j^\pm(z) = \exp\left(\pm \frac{\hbar D_{j,0}}{2}\right) \exp\left(\pm (q - q^{-1}) \sum_{s \geq 1} D_{j,\pm s} z^{\mp s}\right)$$

The relations are

(QL1-gl) For any  $j, j' \in \mathbf{J}$  and  $r, s \in \mathbb{Z}$ ,

$$[D_{j,r}, D_{j',s}] = 0$$

(QL2-gl) For any  $i \in \mathbf{I}$  and  $j \in \mathbf{J}$ ,

$$\Theta_j^\pm(z) E_i(w) \Theta_j^\pm(z)^{-1} = \theta_{c_{ji}}(q^{c_{ji}} z/w) E_i(w)$$

$$\Theta_j^\pm(z)^{-1} F_i(w) \Theta_j^\pm(z) = \theta_{c_{ji}}(q^{c_{ji}} z/w) F_i(w)$$

where  $c_{ji} = -\delta_{ji} + \delta_{j, i+1}$ ,  $\theta_m(\zeta) = \frac{q^m \zeta - 1}{\zeta - q^m}$ , and the right-hand side is expanded in powers of  $z^{\mp 1}$ .<sup>1</sup>

(QL3-gl) For any  $i, i' \in \mathbf{I}$ ,

$$E_i(z) E_{i'}(w) = \theta_{a_{ii'}}(q^{i-i'} z/w) E_{i'}(w) E_i(z)$$

$$F_i(z) F_{i'}(w) = \theta_{a_{ii'}}(q^{i-i'} z/w)^{-1} F_{i'}(w) F_i(z)$$

where  $a_{ii'} = 2\delta_{ii'} - \delta_{|i-i'|,1}$  are the entries of the Cartan matrix of  $\mathfrak{sl}_n$  and the equalities are understood as holding after both side have been multiplied by the denominator of the function  $\theta_m$ .

(QL4-gl) For any  $i, i' \in \mathbf{I}$ ,

$$(q - q^{-1})[E_i(z), F_{i'}(w)] = \delta_{i,i'} \delta(z/w) \left( \frac{\Theta_{i+1}^+(z)}{\Theta_i^+(z)} - \frac{\Theta_{i+1}^-(z)}{\Theta_i^-(z)} \right)$$

where  $\delta(\zeta) = \sum_{r \in \mathbb{Z}} \zeta^r$  is the formal delta function.

(QL5-gl) For any  $i, i' \in \mathbf{I}$  such that  $|i - i'| = 1$ ,

$$E_i(z_1) E_i(z_2) E_{i'}(w) - (q + q^{-1}) E_i(z_1) E_{i'}(w) E_i(z_2) + \\ E_{i'}(w) E_i(z_1) E_i(z_2) + (z_1 \leftrightarrow z_2) = 0$$

$$F_i(z_1) F_i(z_2) F_{i'}(w) - (q + q^{-1}) F_i(z_1) F_{i'}(w) F_i(z_2) + \\ F_{i'}(w) F_i(z_1) F_i(z_2) + (z_1 \leftrightarrow z_2) = 0$$

---

<sup>1</sup>note that the expansions in  $z^{\pm 1}$  are related by the symmetry  $\theta_m(\zeta^{-1}) = \theta_{-m}(\zeta)$ .



For any  $i, i' \in \mathbf{I}$  such that  $|i - i'| \geq 2$ ,

$$E_i(z)E_{i'}(w) = E_{i'}(w)E_i(z)$$

$$F_i(z)F_{i'}(w) = F_{i'}(w)F_i(z)$$

We denote by  $U^0 \subset U_h(L\mathfrak{gl}_n)$  the commutative subalgebra generated by the elements  $D_{j,r}$ .

**7.2. The Yangian  $Y_h(\mathfrak{gl}_n)$ .** The following definition can be found in [16, §3.1].  $Y_h(\mathfrak{gl}_n)$  is the algebra over  $\mathbb{C}[\hbar]$  generated by  $\{e_{i,r}, f_{i,r}, \theta_{j,r}\}_{i \in \mathbf{I}, j \in \mathbf{J}, r \in \mathbb{N}}$ . In order to describe the relations we introduce the following formal power series

$$\theta_j(u) = 1 + \hbar \sum_{r \geq 0} \theta_{j,r} u^{-r-1}$$

$$e_i(u) = \hbar \sum_{r \geq 0} e_{i,r} u^{-r-1}$$

$$f_i(u) = \hbar \sum_{r \geq 0} f_{i,r} u^{-r-1}$$

(Y1-gl) For any  $j, j' \in \mathbf{J}$  and  $r, s \in \mathbb{N}$ ,

$$[\theta_{j,r}, \theta_{j',s}] = 0$$

(Y2-gl) For any  $j \in \mathbf{J}$  and  $i \in \mathbf{I}$ ,

$$(u - v)[\theta_j(u), e_i(v)] = \hbar(\delta_{ji} - \delta_{j,i+1})(e_i(u) - e_i(v))\theta_j(u)$$

$$(u - v)[\theta_j(u), f_i(v)] = \hbar(\delta_{ji} - \delta_{j,i+1})\theta_j(u)(f_i(v) - f_i(u))$$

(Y3-gl) For any  $i \in \mathbf{I}$ ,

$$(u - v)[e_i(u), e_i(v)] = -\hbar(e_i(u) - e_i(v))^2$$

$$(u - v)[f_i(u), f_i(v)] = \hbar(f_i(u) - f_i(v))^2$$

For any  $i \in \mathbf{I} \setminus \{n - 1\}$  and  $r, s \in \mathbb{N}$ ,

$$[e_{i,r+1}, e_{i+1,s}] - [e_{i,r}, e_{i+1,s+1}] = -\hbar e_{i+1,s} e_{i,r}$$

$$[f_{i,r+1}, f_{i+1,s}] - [f_{i,r}, f_{i+1,s+1}] = \hbar f_{i,r} f_{i+1,s}$$

(Y4-gl) For any  $i, i' \in \mathbf{I}$ ,

$$(u - v)[e_i(u), f_{i'}(v)] = \delta_{i,i'} \hbar \left( \frac{\theta_{i+1}(v)}{\theta_i(v)} - \frac{\theta_{i+1}(u)}{\theta_i(u)} \right)$$

(Y5-gl) For any  $i, i' \in \mathbf{I}$  be such that  $|i - i'| = 1$ ,

$$[e_{i,r_1}, [e_{i,r_2}, e_{i',s}]] + [e_{i,r_2}, [e_{i,r_1}, e_{i',s}]] = 0$$

$$[f_{i,r_1}, [f_{i,r_2}, f_{i',s}]] + [f_{i,r_2}, [f_{i,r_1}, f_{i',s}]] = 0$$

For  $i, i' \in \mathbf{I}$  such that  $|i - i'| > 1$ ,

$$[e_{i,r}, e_{i',s}] = 0 = [f_{i,r}, f_{i',s}]$$

The Yangian  $Y_h(\mathfrak{gl}_n)$  is  $\mathbb{N}$ -graded by  $\deg(e_{i,r}) = \deg(f_{i,r}) = \deg(\theta_{j,r}) = r$  and  $\deg(\hbar) = 1$ .

**7.3. Shift homomorphisms.** Let  $Y^0 \subset Y_{\hbar}(\mathfrak{gl}_n)$  be the commutative subalgebra generated by the elements  $\{\theta_{j,r}\}$  and  $Y^+, Y^- \subset Y_{\hbar}(\mathfrak{gl}_n)$  the subalgebras generated by  $Y^0$  and the elements  $\{e_{i,r}\}$  (resp.  $\{f_{i,r}\}$ ),  $i \in \mathbf{I}, r \in \mathbb{N}$ .

For any  $i \in \mathbf{I}$ , define, as in Section 2.6, a  $Y^0$ -linear homomorphism  $\sigma_i^{\pm}$  of  $Y^{\pm}$  by  $e_{i',r} \rightarrow e_{i',r+\delta_{ii'}}$  (resp.  $f_{i',r} \rightarrow f_{i',r+\delta_{ii'}}$ ). The definition of  $\sigma_i^{\pm}$  relies on the PBW theorem for  $Y_{\hbar}(\mathfrak{gl}_n)$ , which is proved in [18].

**7.4. Alternative generators for  $Y^0$ .** Define an alternative family of generators  $\{d_{j,r}\}_{j \in \mathbf{J}, r \in \mathbb{N}}$  of  $Y^0$  by

$$d_j(u) = \hbar \sum_{r \geq 0} d_{j,r} u^{-r-1} = \log(\theta_j(u))$$

Set  $B_j(v) = \hbar \sum_{r \geq 0} d_{j,r} \frac{v^r}{r!} \in Y^0[[v]]$ . The following commutation relations are proved exactly as their counterparts in Lemma 2.9.

**Lemma.** *The following holds for any  $j \in \mathbf{J}$  and  $i \in \mathbf{I}$ ,*

$$\begin{aligned} [B_j(v), e_{i,s}] &= \frac{1 - e^{-c_{ji}\hbar v}}{v} e^{\sigma_i^+ v} e_{i,s} \\ [B_j(v), f_{i,s}] &= -\frac{1 - e^{-c_{ji}\hbar v}}{v} e^{\sigma_i^- v} f_{i,s} \end{aligned}$$

**7.5. The operators  $\lambda_i^{\pm}(\mathbf{v})$ .** Similar to the construction of Section 2.10, we have the following result.

**Proposition.** *There are operators  $\{\lambda_{i;s}^{\pm}\}_{i \in \mathbf{I}, s \in \mathbb{N}}$  on  $Y^0$  such that the following holds.*

(1) *For any  $\xi \in Y^0$ ,*

$$\begin{aligned} e_{i,r}\xi &= \sum_{s \geq 0} \lambda_{i;s}^+(\xi) e_{i,r+s} \\ f_{i,r}\xi &= \sum_{s \geq 0} \lambda_{i;s}^-(\xi) f_{i,r+s} \end{aligned}$$

(2) *The operator  $\lambda_i^{\pm}(v) : Y^0 \rightarrow Y^0[[v]]$  given by*

$$\lambda_i^{\pm}(v)(\xi) = \sum_{s \in \mathbb{N}} \lambda_{i;s}^{\pm}(\xi) v^s$$

*is an algebra homomorphism of degree 0 with respect to the  $\mathbb{N}$ -grading on  $Y^0[[v]]$  extending that on  $Y^0$  by  $\deg(v) = 1$ .*

(3) *The operators  $\lambda_i^{\epsilon}(v)$  and  $\lambda_{i'}^{\epsilon'}(v')$  commute for any  $i, i' \in \mathbf{I}$  and  $\epsilon, \epsilon' \in \{\pm 1\}$ . Moreover,*

$$\lambda_i^+(v) \lambda_i^-(v) = Id$$

(4) *For any  $i \in \mathbf{I}$  and  $j \in \mathbf{J}$ ,*

$$(\lambda_i^{\pm}(v_1) - 1) B_j(v_2) = \pm \frac{e^{-c_{ji}\hbar v_2} - 1}{v_2} e^{v_1 v_2} \quad (7.1)$$

7.6. Let  $\{g_i^\pm(u)\}_{i \in \mathbf{I}}$  be a collection of elements in  $\widehat{Y^0[u]}$ . Define, as in Section 3.1, an assignment  $\Phi : \{E_{i,r}, F_{i,r}, D_{j,r}\} \rightarrow \widehat{Y_h(\mathfrak{gl}_n)}$  by

$$\begin{aligned}\Phi(D_{j,0}) &= \theta_{j,0} \\ \Phi(D_{j,r}) &= \frac{B_j(r)}{q - q^{-1}} \text{ for } r \neq 0 \\ \Phi(E_{i,k}) &= e^{k\sigma_i^+} g_i^+(\sigma_i^+) e_{i,0} \\ \Phi(F_{i,k}) &= e^{k\sigma_i^-} g_i^-(\sigma_i^-) f_{i,0}\end{aligned}$$

and denote the restriction of  $\Phi$  to  $U^0$  by  $\Phi^0 : U^0 \rightarrow \widehat{Y^0}$ .

**Theorem.** *The assignment  $\Phi$  extends to an algebra homomorphism if and only if the following conditions hold.*

(A) *For any  $i, i' \in \mathbf{I}$ ,*

$$g_i^+(u) \lambda_i^+(u) (g_{i'}^-(v)) = g_{i'}^-(v) \lambda_{i'}^-(v) (g_i^+(u))$$

(B) *For any  $i \in \mathbf{I}$  and  $k \in \mathbb{Z}$ ,*

$$e^{ku} g_i^+(v) \lambda_i^+(v) (g_i^-(v))|_{v^m = \xi_{i,m}} = \Phi^0 \left( \frac{P_{i,k}^+ - P_{i,k}^-}{q - q^{-1}} \right)$$

where

$$\xi_i(u) = 1 + \hbar \sum_{r \geq 0} \xi_{i,r} u^{-r-1} = \theta_{i+1}(u) \theta_i(u)^{-1} \in Y^0[[u^{-1}]]$$

and

$$P_i^\pm(z) = \sum_{s \geq 0} P_{i,\pm s}^\pm z^{\mp s} = \Theta_{i+1}^\pm(z) \Theta_i^\pm(z)^{-1} \in U^0[[z^{\mp 1}]]$$

(C0) *For any  $i, i' \in \mathbf{I}$  such that  $|i - i'| > 1$ ,*

$$g_i^\pm(u) \lambda_i^\pm(u) (g_{i'}^\pm(v)) = g_{i'}^\pm(v) \lambda_{i'}^\pm(v) (g_i^\pm(u))$$

(C1) *For any  $i \in \mathbf{I}$*

$$g_i^\pm(u) \lambda_i^\pm(u) (g_i^\pm(v)) \frac{e^u - e^{v \pm \hbar}}{u - v \mp \hbar} = g_i^\pm(v) \lambda_i^\pm(v) (g_i^\pm(u)) \frac{e^v - e^{u \pm \hbar}}{v - u \mp \hbar}$$

(C2) *For any  $i \in \mathbf{I} \setminus \{n-1\}$ ,*

$$g_i^\pm(u) \lambda_i^\pm(u) (g_{i+1}^\pm(v)) \left( \frac{e^u - e^v}{u - v} \right)^{\pm 1} = g_{i+1}^\pm(v) \lambda_{i+1}^\pm(v) (g_i^\pm(u)) \left( \frac{e^{u-\hbar/2} - e^{v+\hbar/2}}{u - v - \hbar} \right)^{\pm 1}$$

The proof of Theorem 7.6 is given in 7.7–7.11. It follows the same lines as that of Theorem 3.3, with the exception of the  $q$ -Serre relations (QL5-gl) which are proved directly.

7.7. A proof similar to that of Lemmas 3.4 and 3.5 yields the following

- (1)  $\Phi$  is compatible with the relation (QL4-gl) if, and only if (A) and (B) hold.
- (2)  $\Phi$  is compatible with the relation (QL3-gl) if, and only if (C0)–(C2) hold.

By virtue of condition (C0),  $\Phi$  is compatible with the the  $q$ -Serre relations (QL5-gl) whenever  $|i - i'| > 1$ . We therefore need only consider the case  $|i - i'| = 1$ . We shall in fact restrict to  $i' = i + 1$  since the case  $i' = i - 1$  is dealt with similarly.

7.8. The essential ingredient is the following analogue of Lemma A.4. We leave it to the reader to carry out the construction of the auxiliary algebra  $\overline{Y}$  (see § A.2), the operators  $\overline{\sigma}_{i,(1)}, \overline{\sigma}_{i,(2)}$  and  $\overline{\sigma}_{i'}$  on  $\overline{Y}_{2\alpha_i + \alpha_{i'}}$  (§ A.3) and the map  $p_{ii'} : \overline{Y}_{2\alpha_i + \alpha_{i'}} \rightarrow Y_{\hbar}(\mathfrak{gl}_n)$ .

**Lemma.** *The kernel of  $p_{ii'}$  is the  $\mathbb{C}[\hbar]$ -linear span of the following elements*

- (1) For any  $A(u_1, u_2, v) \in \overline{Y}^0[u_1, u_2, v]$ 

$$A(\overline{\sigma}_{i,(1)}, \overline{\sigma}_{i,(2)}, \overline{\sigma}_{i'}) ((\overline{\sigma}_{i,(2)} - \overline{\sigma}_{i'}) \overline{e}_{i,0}^2 \overline{e}_{i',0} - (\overline{\sigma}_{i,(2)} - \overline{\sigma}_{i'} - \hbar) \overline{e}_{i,0} \overline{e}_{i',0} \overline{e}_{i,0})$$

$$A(\overline{\sigma}_{i,(1)}, \overline{\sigma}_{i,(2)}, \overline{\sigma}_{i'}) ((\overline{\sigma}_{i,(1)} - \overline{\sigma}_{i'}) \overline{e}_{i,0} \overline{e}_{i',0} \overline{e}_{i,0} - (\overline{\sigma}_{i,(1)} - \overline{\sigma}_{i'} - \hbar) \overline{e}_{i',0}^2 \overline{e}_{i,0})$$
- (2) For any  $B(u_1, u_2, v) = B(u_2, u_1, v) \in \overline{Y}^0[u_1, u_2, v]$ 

$$B(\overline{\sigma}_{i,(1)}, \overline{\sigma}_{i,(2)}, \overline{\sigma}_{i'}) (\overline{\sigma}_{i,(1)} - \overline{\sigma}_{i,(2)} - \hbar) \overline{e}_{i,0}^2 \overline{e}_{i',0}$$

$$B(\overline{\sigma}_{i,(1)}, \overline{\sigma}_{i,(2)}, \overline{\sigma}_{i'}) (\overline{\sigma}_{i,(1)} - \overline{\sigma}_{i,(2)} - \hbar) \overline{e}_{i',0} \overline{e}_{i,0}^2$$
- (3) For any  $B(u_1, u_2, v) = B(u_2, u_1, v) \in \overline{Y}^0[u_1, u_2, v]$ 

$$B(\overline{\sigma}_{i,(1)}, \overline{\sigma}_{i,(2)}, \overline{\sigma}_{i'}) (\overline{e}_{i,0}^2 \overline{e}_{i',0} - 2 \overline{e}_{i,0} \overline{e}_{i',0} \overline{e}_{i,0} + \overline{e}_{i',0} \overline{e}_{i,0}^2)$$

**Corollary.** *The kernel of  $p_{ii'}$  is stable under the action of  $A(\overline{\sigma}_{i,(1)}, \overline{\sigma}_{i,(2)}, \overline{\sigma}_{i'})$ , for any  $A(u_1, u_2, v) = A(u_2, u_1, v) \in \overline{Y}^0[u_1, u_2, v]$ .*

**Remark.** In the next sections we use the following convention for notational convenience: for each  $\overline{X} \in \overline{Y}_{2\alpha_i + \alpha_{i'}}$  and  $X = p_{ii'}(\overline{X})$ , we set

$$A(\sigma_{i,1}, \sigma_{i,2}, \sigma_{i'})(X) = p_{i,i'}(A(\overline{\sigma}_{i,(1)}, \overline{\sigma}_{i,(2)}, \overline{\sigma}_{i'}) (\overline{X}))$$

7.9. We shall only prove the  $q$ -Serre relations for the case of the  $E$ 's and consequently drop the superscript  $+$ . We need to show that the following holds for any  $k_1, k_2, l \in \mathbb{Z}$

$$\begin{aligned} \Phi(E_{i,k_1}) \Phi(E_{i,k_2}) \Phi(E_{j,l}) - (q + q^{-1}) \Phi(E_{i,k_1}) \Phi(E_{j,l}) \Phi(E_{i,k_2}) \\ + \Phi(E_{j,l}) \Phi(E_{i,k_1}) \Phi(E_{i,k_2}) + (k_1 \leftrightarrow k_2) = 0 \end{aligned}$$

As in § A.5, an application of Corollary 7.8 shows that this reduces to showing that

$$\Phi(E_{i,0})^2 \Phi(E_{j,0}) - (q + q^{-1}) \Phi(E_{i,0}) \Phi(E_{j,0}) \Phi(E_{i,0}) + \Phi(E_{j,0}) \Phi(E_{i,0})^2 = 0$$

7.10. With Corollary 7.8 in mind, we seek to factor a common symmetric function out of each of the above summands. This is achieved by the following result.

**Lemma.** *There exists  $H(u_1, u_2, v) \in \overline{Y}^0[[u_1, u_2, v]]$  symmetric in  $u_1 \leftrightarrow u_2$ , such that*

$$\begin{aligned}\Phi(E_{i,0})^2\Phi(E_{i',0}) &= H(\sigma_{i,1}, \sigma_{i,2}, \sigma_{i'})\mathcal{P}_0(\sigma_{i,1}, \sigma_{i,2}, \sigma_{i'})e_{i,0}e_{i',0}e_{i,0} \\ \Phi(E_{i,0})\Phi(E_{i',0})\Phi(E_{i,0}) &= H(\sigma_{i,1}, \sigma_{i,2}, \sigma_{i'})\mathcal{P}_1(\sigma_{i,1}, \sigma_{i,2}, \sigma_{i'})e_{i,0}e_{i',0}e_{i,0} \\ \Phi(E_{i',0})\Phi(E_{i,0})^2 &= H(\sigma_{i,1}, \sigma_{i,2}, \sigma_{i'})\mathcal{P}_2(\sigma_{i,1}, \sigma_{i,2}, \sigma_{i'})e_{i',0}e_{i,0}e_{i,0}\end{aligned}$$

where  $\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2 \in \mathbb{C}[[u_1, u_2, v]]$  are given in terms of the function

$$P(x, y) = \frac{e^x - e^y}{x - y} \in \mathbb{C}[[x, y]]^{\mathfrak{S}_2}$$

by

$$\begin{aligned}\mathcal{P}_0 &= P(u_1 + \hbar, u_2)P(u_1 - \hbar/2, v + \hbar/2)P(u_2 - \hbar/2, v + \hbar/2) \\ \mathcal{P}_1 &= P(u_1 + \hbar, u_2)P(u_1 - \hbar/2, v + \hbar/2)P(u_2, v) \\ \mathcal{P}_2 &= P(u_1 + \hbar, u_2)P(u_1, v)P(u_2, v)\end{aligned}$$

PROOF. Define  $G_{ab}(x, y) \in \overline{Y}^0[[x, y]]$  by  $\lambda_a(x)(g_b(y)) = g_b(y)G_{ab}(x, y)$ . Then, in obvious notation,

$$\begin{aligned}\Phi(E_{a,0})\Phi(E_{b,0})\Phi(E_{c,0}) &= g_a(\sigma_a)e_{a,0}g_b(\sigma_b)e_{b,0}g_c(\sigma_c)e_{c,0} \\ &= g_a(\sigma_a)\lambda_a(\sigma_a)(g_b(\sigma_b))\lambda_a(\sigma_a) \circ \lambda_b(\sigma_b)(g_c(\sigma_c))e_{a,0}e_{b,0}e_{c,0} \\ &= g_a(\sigma_a)g_b(\sigma_b)g_c(\sigma_b)G_{ab}(\sigma_a, \sigma_b)G_{ac}(\sigma_a, \sigma_c)\lambda_a(\sigma_a)(G_{bc}(\sigma_b, \sigma_c))e_{a,0}e_{b,0}e_{c,0}\end{aligned}$$

We record for later use the symmetry in the interchange  $a \leftrightarrow b$  of the term

$$\begin{aligned}G_{ac}(\sigma_a, \sigma_c)\lambda_a(\sigma_a)(G_{bc}(\sigma_b, \sigma_c)) &= \lambda_a(\sigma_a) \circ \lambda_b(\sigma_b)(g_c(\sigma_c))/g_c(\sigma_c) \\ &= G_{bc}(\sigma_b, \sigma_c)\lambda_b(\sigma_b)(G_{ac}(\sigma_a, \sigma_c))\end{aligned}\tag{7.2}$$

where the second equality follows from the commutativity of  $\lambda_a(\sigma_a)$  and  $\lambda_b(\sigma_b)$ .

Set now  $F = g_i(\sigma_{i,1})g_i(\sigma_{i,2})g_{i'}(\sigma_{i'}) \in \overline{Y}^0[[\sigma_{i,1}, \sigma_{i,2}, \sigma_{i'}]]^{\mathfrak{S}_2}$ . Then, the above yields

$$\begin{aligned}\Phi(E_{i,0})^2\Phi(E_{i',0}) &= F G_{ii}(\sigma_{i,1}, \sigma_{i,2})G_{ii'}(\sigma_{i,1}, \sigma_{i'})\lambda_i(\sigma_{i,1})(G_{ii'}(\sigma_{i,2}, \sigma_{i'}))e_{i,0}^2e_{i',0} \\ \Phi(E_{i,0})\Phi(E_{i',0})\Phi(E_{i,0}) &= F G_{ii'}(\sigma_{i,1}, \sigma_{i'})G_{ii}(\sigma_{i,1}, \sigma_{i,2})\lambda_i(\sigma_{i,1})(G_{i'i}(\sigma_{i'}, \sigma_{i,2}))e_{i,0}e_{i',0}e_{i,0} \\ \Phi(E_{i',0})\Phi(E_{i,0})^2 &= F G_{i'i}(\sigma_{i'}, \sigma_{i,1})G_{i'i}(\sigma_{i'}, \sigma_{i,2})\lambda_{i'}(\sigma_{i'})(G_{ii}(\sigma_{i,1}, \sigma_{i,2}))e_{i',0}e_{i,0}^2\end{aligned}$$

We claim that  $G_{ii}(u_1, u_2) = \overline{G}_{ii}(u_1, u_2)P(u_1 + \hbar, u_2)$ , where  $\overline{G}$  is symmetric in  $u_1, u_2$ . Indeed, by condition (C1)

$$G_{ii}(u_1, u_2)P(u_1, u_2 + \hbar) = G_{ii}(u_2, u_1)P(u_2, u_1 + \hbar)$$

whence the result with  $\overline{G}_{ii}(u_1, u_2) = G_{ii}(u_1, u_2)/P(u_1 + \hbar, u_2)$ . It follows that

$$\Phi(E_{i,0})^2\Phi(E_{i',0}) = \overline{H}(\sigma_{i,1}, \sigma_{i,2}, \sigma_{i'})P(\sigma_{i,1} + \hbar, \sigma_{i,2})e_{i,0}^2e_{i',0}$$

where

$$\overline{H}(u_1, u_2, v) = F(u_1, u_2, v)\overline{G}_{ii}(u_1, u_2)G_{ii'}(u_1, v)\lambda_i(u_1)(G_{ii'}(u_2, v)) \in Y^0[[u_1, u_2, v]]$$

is symmetric in  $u_1, u_2$  by (7.2).

Next, assuming that  $i' = i + 1$ , condition (C2) yields

$$G_{ii'}(u, v)P(u, v) = G_{i'i}(v, u)P(u - \hbar/2, v + \hbar/2)$$

so that

$$\begin{aligned} \Phi(E_{i,0})\Phi(E_{i',0})\Phi(E_{i,0}) = \\ \overline{H}(\sigma_{i,1}, \sigma_{i,2}, \sigma_{i'})P(\sigma_{i,1} + \hbar, \sigma_{i,2}) \frac{P(\sigma_{i,2}, \sigma_{i'})}{P(\sigma_{i,2} - \hbar/2, \sigma_{i'} + \hbar/2)} e_{i,0}e_{i',0}e_{i,0} \end{aligned}$$

Finally, using (7.2) again, with  $a = i, b = i', c = i$  and the previous calculation yields

$$\begin{aligned} \Phi(E_{i',0})\Phi(E_{i,0})^2 = \\ \overline{H}(\sigma_{i,1}, \sigma_{i,2}, \sigma_{i'})P(\sigma_{i,1} + \hbar, \sigma_{i,2}) \frac{P(\sigma_{i,1}, \sigma_{i'})}{P(\sigma_{i,1} - \hbar/2, \sigma_{i'} + \hbar/2)} \frac{P(\sigma_{i,2}, \sigma_{i'})}{P(\sigma_{i,2} - \hbar/2, \sigma_{i'} + \hbar/2)} e_{i',0}e_{i,0}^2 \end{aligned}$$

as claimed.  $\square$

7.11. By Lemma 7.10 and Corollary 7.8, we are reduced to proving the following

$$\begin{aligned} \mathcal{S}^q = \mathcal{P}_0(\sigma_{i,1}, \sigma_{i,2}, \sigma_{i'})e_{i,0}^2e_{i',0} - (q + q^{-1})\mathcal{P}_1(\sigma_{i,1}, \sigma_{i,2}, \sigma_{i'})e_{i,0}e_{i',0}e_{i,0} \\ + \mathcal{P}_2(\sigma_{i,1}, \sigma_{i,2}, \sigma_{i'})e_{i',0}e_{i,0}^2 = 0 \end{aligned}$$

**Step 1.** We first observe that

$$P(u_1 + \hbar, u_2) - \frac{1 + e^\hbar}{2}P(u_1, u_2) \in (u_1 - u_2 - \hbar)\mathbb{C}[[\hbar, u_1, u_2]]^{\mathfrak{S}_2}$$

This allows us to use (2) of Lemma 7.8 to obtain

$$\mathcal{S}^q = \mathcal{P}'_0(\sigma_{i,1}, \sigma_{i,2}, \sigma_{i'})e_{i,0}^2e_{i',0} - 2\mathcal{P}'_1(\sigma_{i,1}, \sigma_{i,2}, \sigma_{i'})e_{i,0}e_{i',0}e_{i,0} + \mathcal{P}'_2(\sigma_{i,1}, \sigma_{i,2}, \sigma_{i'})e_{i',0}e_{i,0}^2$$

where

$$\begin{aligned} \mathcal{P}'_0 &= e^{\hbar/2}P(u_1, u_2)P(u_1 - \hbar/2, v + \hbar/2)P(u_2 - \hbar/2, v + \hbar/2) \\ \mathcal{P}'_1 &= P(u_1 + \hbar, u_2)P(u_1 - \hbar/2, v + \hbar/2)P(u_2, v) = \mathcal{P}_1 \\ \mathcal{P}'_2 &= e^{\hbar/2}P(u_1, u_2)P(u_1, v)P(u_2, v) \end{aligned}$$

**Step 2.** We use next (3) of Lemma 7.8 with  $B = \mathcal{P}'_2$  to get

$$\mathcal{S}^q = (\mathcal{P}'_0 - \mathcal{P}'_2)e_{i,0}^2e_{i',0} - 2(\mathcal{P}'_1 - \mathcal{P}'_2)e_{i,0}e_{i',0}e_{i,0}$$

One can easily check that  $\mathcal{P}'_1 - \mathcal{P}'_2$  is divisible by  $u_2 - v - \hbar$ , which together with

(1) of Lemma 7.8, with  $A = 2\frac{\mathcal{P}'_1 - \mathcal{P}'_2}{u_2 - v - \hbar}$ , yields

$$\mathcal{S}^q = \left( \mathcal{P}'_0 - \mathcal{P}'_2 - 2\frac{\mathcal{P}'_1 - \mathcal{P}'_2}{u_2 - v - \hbar}(u_2 - v) \right) e_{i,0}^2e_{i',0}$$

**Step 3.** Finally we can directly verify that the function

$$\mathcal{F} = \mathcal{P}'_0 - \mathcal{P}'_2 - 2\frac{\mathcal{P}'_1 - \mathcal{P}'_2}{u_2 - v - \hbar}(u_2 - v)$$

is divisible by  $u_1 - u_2 - \hbar$ . Moreover the quotient  $\frac{\mathcal{F}}{u_1 - u_2 - \hbar}$  is symmetric in  $u_1, u_2$ . This allows us to use (2) of Lemma 7.8 to conclude that  $\mathcal{S}^q = 0$ .

**7.12. The variety  $\mathcal{F}$ .** Fix  $d \in \mathbb{N}$ , let

$$\mathcal{F} = \left\{ 0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n = \mathbb{C}^d \right\}$$

denote the variety of  $n$ -step flags in  $\mathbb{C}^d$  and  $T^*\mathcal{F}$  its cotangent bundle. We describe below the  $GL_d(\mathbb{C}) \times \mathbb{C}^\times$ -equivariant  $K$ -theory and Borel–Moore homology of  $T^*\mathcal{F}$  following [10, 23].

The connected components of  $\mathcal{F}$  are parametrised by the set  $\mathcal{P}$  of partitions of  $[1, d]$  into  $n$  intervals, *i.e.*,

$$\mathcal{P} = \{\mathbf{d} = (0 = d_0 \leq d_1 \leq \cdots \leq d_n = d)\}$$

where  $\mathbf{d} \in \mathcal{P}$  labels the component  $\mathcal{F}_{\mathbf{d}}$  consisting of flags such that  $\dim V_k = d_k$ . The symmetric group  $\mathfrak{S}_d$  acts on the rings

$$\begin{aligned} S &= \mathbb{C}[q^{\pm 1}, X_1^{\pm 1}, \dots, X_d^{\pm 1}] \\ R &= \mathbb{C}[\hbar, x_1, \dots, x_d] \end{aligned}$$

by permuting the variables  $X_1, \dots, X_d$  and  $x_1, \dots, x_d$  respectively. For any  $\mathbf{d} \in \mathcal{P}$ , denote by

$$\mathfrak{S}(\mathbf{d}) = \mathfrak{S}_{d_1 - d_0} \times \cdots \times \mathfrak{S}_{d_n - d_{n-1}} \subset \mathfrak{S}_d$$

the subgroup preserving the corresponding partition. Then, the following holds

$$\begin{aligned} K^{GL_d(\mathbb{C}) \times \mathbb{C}^\times}(T^*\mathcal{F}) &\cong \bigoplus_{\mathbf{d} \in \mathcal{P}} S^{\mathfrak{S}(\mathbf{d})} \\ H^{GL_d(\mathbb{C}) \times \mathbb{C}^\times}(T^*\mathcal{F}) &\cong \bigoplus_{\mathbf{d} \in \mathcal{P}} R^{\mathfrak{S}(\mathbf{d})} \end{aligned}$$

where  $K^{\mathbb{C}^\times}(pt) = \mathbb{C}[q, q^{-1}]$  and  $H^{\mathbb{C}^\times}(pt) = \mathbb{C}[\hbar]$ .

**7.13.** For any partition  $\mathbf{d} \in \mathcal{P}$  and  $i \in \mathbf{I}$ , set

$$\mathbf{d}_i^\pm = (0 = d_0 \leq \cdots \leq d_{i-1} \leq d_i \pm 1 \leq d_{i+1} \leq \cdots \leq d_n = d)$$

if the right-hand side makes sense as a partition.

If  $\mathbf{d}, \mathbf{d}' \in \mathcal{P}$  are two partitions, and  $P$  is one of the rings  $R, S$ , we denote by  $\sigma(\mathbf{d}, \mathbf{d}')$  the symmetrisation operator

$$\sigma(\mathbf{d}, \mathbf{d}') : P^{\mathfrak{S}(\mathbf{d}) \cap \mathfrak{S}(\mathbf{d}')} \rightarrow P^{\mathfrak{S}(\mathbf{d}')}, \quad \sigma(\mathbf{d}, \mathbf{d}')(p) = \sum_{\tau \in \mathfrak{S}(\mathbf{d}) / \mathfrak{S}(\mathbf{d}) \cap \mathfrak{S}(\mathbf{d}')} \tau(p)$$

7.14.  $U_h(L\mathfrak{gl}_n)$ –**action** [10, 23]. Consider the following operators acting on

$$S(\mathcal{P}) = \bigoplus_{\mathbf{d} \in \mathcal{P}} S^{\mathfrak{S}(\mathbf{d})}$$

(1) For any  $j \in \mathbf{J}$ ,  $\Psi_U(\Theta_j^\pm(z))$  acts on  $S^{\mathfrak{S}(\mathbf{d})}$  as multiplication by

$$\prod_{k=1}^{d_j-1} \frac{qz - q^{-1}X_k}{z - X_k} \prod_{k=d_j+1}^d \frac{z - X_k}{q^{-1}z - qX_k} \in S^{\mathfrak{S}(\mathbf{d})}[[z^{\mp 1}]]$$

(2) For any  $i \in \mathbf{I}$ , the operators

$$\begin{aligned} \Psi_U(E_i(z)) &: S^{\mathfrak{S}(\mathbf{d})} \rightarrow S^{\mathfrak{S}(\mathbf{d}_i^+)}[[z, z^{-1}]] \\ \Psi_U(F_i(z)) &: S^{\mathfrak{S}(\mathbf{d})} \rightarrow S^{\mathfrak{S}(\mathbf{d}_i^-)}[[z, z^{-1}]] \end{aligned}$$

act by 0 if  $\mathbf{d}_i^\pm \notin \mathcal{P}$ , and by

$$\begin{aligned} \Psi_U(E_i(z))p &= \sigma(\mathbf{d}, \mathbf{d}_i^+) \left( p\delta(X_{d_i+1}/z) \prod_{k \in I_i} \frac{qX_{d_i+1} - q^{-1}X_k}{X_{d_i+1} - X_k} \right) \\ \Psi_U(F_i(z))p &= \sigma(\mathbf{d}, \mathbf{d}_i^-) \left( p\delta(X_{d_i+1}/z) \prod_{k \in I_{i+1}} \frac{q^{-1}X_{d_i} - qX_k}{X_{d_i} - X_k} \right) \end{aligned}$$

otherwise, where  $I_i$  is the interval  $[d_{i-1} + 1, \dots, d_i]$ .

The following result is due to Ginzburg and Vasserot and is proved in [23, §2.2]

**Theorem.** *The assignment  $\Psi_U$  extends to an algebra homomorphism*

$$\Psi_U : U_h(L\mathfrak{gl}_n) \rightarrow \text{End}_{\mathbb{C}[q, q^{-1}]}(S(\mathcal{P}))$$

7.15.  $Y_h(\mathfrak{gl}_n)$ –**action**. Consider the following operators acting on

$$R(\mathcal{P}) = \bigoplus_{\mathbf{d} \in \mathcal{P}} R^{\mathfrak{S}(\mathbf{d})}$$

(1) For any  $j \in \mathbf{J}$ ,  $\Psi_Y(\theta_j(u))$  acts on  $R^{\mathfrak{S}(\mathbf{d})}$  as multiplication by

$$\prod_{k=1}^{d_j-1} \frac{u - x_k + \hbar}{u - x_k} \prod_{k=d_j+1}^d \frac{u - x_k}{u - x_k - \hbar} \in R^{\mathfrak{S}(\mathbf{d})}[[u^{-1}]]$$

(2) For any  $i \in \mathbf{I}$ ,

$$\begin{aligned} \Psi_Y(e_i(u)) &: R^{\mathfrak{S}(\mathbf{d})} \rightarrow R^{\mathfrak{S}(\mathbf{d}_i^+)}[[u^{-1}]] \\ \Psi_Y(f_i(u)) &: R^{\mathfrak{S}(\mathbf{d})} \rightarrow R^{\mathfrak{S}(\mathbf{d}_i^-)}[[u^{-1}]] \end{aligned}$$



act as zero if  $\mathbf{d}_i^\pm \notin \mathcal{P}$ , and by

$$\begin{aligned}\Psi_Y(e_i(u))p &= \hbar\sigma(\mathbf{d}, \mathbf{d}_i^+) \left( \frac{p}{u - x_{d_i+1}} \prod_{k \in I_i} \frac{x_{d_i+1} - x_k + \hbar}{x_{d_i+1} - x_k} \right) \\ \Psi_Y(f_i(u))p &= \hbar\sigma(\mathbf{d}, \mathbf{d}_i^-) \left( \frac{p}{u - x_{d_i}} \prod_{k \in I_{i+1}} \frac{x_{d_i} - x_k - \hbar}{x_{d_i} - x_k} \right)\end{aligned}$$

otherwise.

The following result is proved in a similar way to Theorem 7.14

**Proposition.** *The assignment  $\Psi_Y$  extends to an algebra homomorphism*

$$\Psi_Y : Y_\hbar(\mathfrak{gl}_n) \rightarrow \text{End}_{\mathbb{C}[\hbar]}(R(\mathcal{P}))$$

**Remark.** The above formulae are degenerations of those of the previous section obtained by setting  $z = e^{tu}$ ,  $q = e^{t\hbar/2}$ ,  $X_k = e^{tx_k}$  and letting  $t \rightarrow 0$ .

7.16.

**Lemma.** *The homomorphism  $\Psi_Y$  maps the center  $\mathcal{Z}$  of  $Y_\hbar(\mathfrak{gl}_n)$  surjectively to  $\mathbb{C}[\hbar, x_1, \dots, x_d]^{\mathfrak{S}_d}$ . In particular, there exists an element*

$$\Delta(u) = 1 + \hbar \sum_{r \geq 0} \Delta_r u^{-r-1} \in \mathcal{Z}[[u^{-1}]]$$

such that

$$\Psi_Y(\Delta(u)) = \prod_{k=1}^d \frac{u - x_k - \hbar}{u - x_k}$$

PROOF. By [16, Cor. 1.11.8],  $\mathcal{Z}$  is generated by the coefficients of the element

$$qdet(u) = \theta_1(u)\theta_2(u - \hbar) \cdots \theta_n(u - (n-1)\hbar) \in Y_\hbar(\mathfrak{gl}_n)[[u^{-1}]]$$

It readily follows from 7.15 that

$$\Psi_Y(qdet(u)) = \prod_{k=1}^d \frac{u - x_k}{u - x_k - (n-1)\hbar}$$

By (2.6),  $L(v) = B(\log(qdet(u))) \in \mathcal{Z}[[v]]$  therefore satisfies

$$\Psi_Y(L(v)) = \sum_{k=1}^d \frac{e^{(x_k + (n-1)\hbar)v} - e^{x_k v}}{v} = \sum_{r \geq 1} \left( p_r(\{x_k + (n-1)\hbar\}) - p_r(\{x_k\}) \right) \frac{v^{r-1}}{r!}$$

which yields the surjectivity since the power sums  $p_r(x_1, \dots, x_d) = \sum_k x_k^r$  generate  $\mathbb{C}[x_1, \dots, x_d]^{\mathfrak{S}_d}$ .  $\square$

7.17. We will need the following

**Lemma.** *For any  $i \in \mathbf{I}$ , there exists  $\mathrm{Td}_i^\pm(v) = \sum_{r \geq 0} \mathrm{Td}_{i,r}^\pm v^r \in \widehat{Y^0[v]}$  such that*

$$\begin{aligned}\Psi_Y(\mathrm{Td}_i^+(v)) &= \prod_{k \in I_i} \frac{v - x_k}{1 - e^{-v+x_k}} \frac{1 - e^{-v+x_k-\hbar}}{v - x_k + \hbar} \\ \Psi_Y(\mathrm{Td}_i^-(v)) &= \prod_{k \in I_{i+1}} \frac{v - x_k}{1 - e^{-v+x_k}} \frac{1 - e^{-v+x_k+\hbar}}{v - x_k - \hbar}\end{aligned}$$

The proof of this lemma is given in Section 7.20.

7.18. **A compatible assignment.** Let  $\Phi : \{E_{i,0}, F_{i,0}, D_{j,r}\}_{i \in \mathbf{I}, j \in \mathbf{J}, r \in \mathbb{N}} \rightarrow \widehat{Y_\hbar(\mathfrak{gl}_n)}$  be the assignment defined by

$$\begin{aligned}\Phi(D_{j,0}) &= \theta_{j,0} \\ \Phi(D_{j,r}) &= \left. \frac{B_j(v)}{q - q^{-1}} \right|_{v=r} \\ \Phi(E_{i,0}) &= \sum_{s \geq 0} e_{i,s} \mathrm{Td}_{i,s}^+ q^{-\Delta_0 - \theta_{i,0}} \\ \Phi(F_{i,0}) &= \sum_{s \geq 0} f_{i,s} \mathrm{Td}_{i,s}^- q^{\Delta_0 + \theta_{i+1,0}}\end{aligned}$$

where  $\Delta_0$  is given by Lemma 7.16. Extend  $\Phi$  to the generators  $E_{i,r}, F_{i,r}$ ,  $r \in \mathbb{Z}$  by defining, as in Section 3.2,

$$\mathrm{Td}_i^{\pm, (r)}(v) = \sum_{m \geq 0} \mathrm{Td}_{i,m}^{\pm, (r)} v^m = e^{rv} \mathrm{Td}_i^\pm(v) \in \widehat{Y^0[[v]]}$$

and setting

$$\begin{aligned}\Phi(E_{i,r}) &= \sum_{s \geq 0} e_{i,s} \mathrm{Td}_{i,s}^{+, (r)} q^{-\Delta_0 - \theta_{i,0}} \\ \Phi(F_{i,r}) &= \sum_{s \geq 0} f_{i,s} \mathrm{Td}_{i,s}^{-, (r)} q^{\Delta_0 + \theta_{i+1,0}}\end{aligned}$$

7.19. Let  $\widehat{R(\mathcal{P})}$  be the completion with respect to the  $\mathbb{N}$ -grading given by  $\deg(x_k) = \deg(\hbar) = 1$ . Define an algebra homomorphism  $\mathrm{ch} : S(\mathcal{P}) \rightarrow \widehat{R(\mathcal{P})}$  mapping each  $S^{\mathfrak{S}(\mathbf{d})}$  to  $\widehat{R^{\mathfrak{S}(\mathbf{d})}}$  by

$$q^2 \mapsto e^{\hbar} \quad \text{and} \quad X_k \mapsto e^{x_k}$$

**Theorem.** *The assignment  $\Phi$  above intertwines the geometric realisations of  $U_\hbar(L\mathfrak{gl}_n)$  and  $Y_\hbar(\mathfrak{gl}_n)$  on  $S(\mathcal{P})$  and  $\widehat{R(\mathcal{P})}$  respectively. Thus, the following holds for any  $X \in U_\hbar(L\mathfrak{gl}_n)$  and  $p \in S(\mathcal{P})$ .*

$$\mathrm{ch}(X \cdot p) = \Phi(X) \cdot \mathrm{ch}(p)$$

PROOF. Consider first the case  $X = D_{j,r}$ ,  $j \in \mathbf{J}$ ,  $r \in \mathbb{Z}$ . By definition of  $\Psi_U$  and  $\Psi_Y$ ,  $D_{j,0} = 2/\hbar \log(\Theta_{j,0})$  and  $\theta_{j,0}$  act on  $S^{\mathfrak{S}(\mathbf{d})}$  and  $R^{\mathfrak{S}(\mathbf{d})}$  respectively as multiplication by  $d - (d_j - d_{j-1})$ . Further, (2.6) yields

$$\Psi_Y(B_j(v)) = \frac{1}{v} \left( (1 - e^{-\hbar v}) \sum_{k=1}^{d_{j-1}} e^{x_k v} + (e^{\hbar v} - 1) \sum_{k=d_j+1}^d e^{x_k v} \right)$$

Similarly, taking log in

$$\Psi_U \left( \exp \left( (q - q^{-1}) \sum_{s \geq 1} D_{j,s} z^{-s} \right) \right) = \prod_{k=1}^{d_{j-1}} \frac{z - q^{-2} X_k}{z - X_k} \prod_{k=d_j+1}^d \frac{z - X_k}{z - q^2 X_k}$$

yields

$$\Psi_U((q - q^{-1})D_{j,r}) = \frac{1}{r} \left( (1 - q^{-2r}) \sum_{k=1}^{d_{j-1}} X_k^r + (q^{2r} - 1) \sum_{k=d_j+1}^d X_k^r \right)$$

Thus,  $\text{ch}(\Psi_U(D_{j,r})p) = \Psi_Y(B_j(r)p/(q - q^{-1}))$  for any  $p \in S^{\mathfrak{S}(\mathbf{d})}$ .

We turn next to  $X = E_{i,0}$ . Let  $P \in S^{\mathfrak{S}(\mathbf{d})}$  and set  $p = \text{ch}(\pi) \in R^{\mathfrak{S}(\mathbf{d})}$ . Since  $\Delta_0$  acts on  $R^{\mathfrak{S}(\mathbf{d})}$  as multiplication by  $-d$  by Lemma 7.16, and  $\theta_{j,0}$  acts as multiplication by  $d - (d_j - d_{j-1})$ , we get

$$\begin{aligned} \Phi(E_{i,0})(p) &= \sum_{s \geq 0} e_{i,s} \left( \text{Td}_{i,s}^+ p \right) q^{d_j - d_{j-1}} \\ &= \sigma(\mathbf{d}, \mathbf{d}_i^+) \left( \sum_{s \geq 0} \text{Td}_{i,s}^+ x_{d_i+1}^s p \prod_{k \in I_i} \frac{x_{d_i+1} - x_k + \hbar}{x_{d_i+1} - x_k} \right) q^{d_j - d_{j-1}} \\ &= \sigma(\mathbf{d}, \mathbf{d}_i^+) \left( \text{Td}_i^+(x_{d_i+1}) p \prod_{k \in I_i} \frac{x_{d_i+1} - x_k + \hbar}{x_{d_i+1} - x_k} \right) q^{d_j - d_{j-1}} \\ &= \sigma(\mathbf{d}, \mathbf{d}_i^+) \left( p \prod_{k \in I_i} \frac{e^{x_{d_i+1}} - e^{x_k - \hbar}}{e^{x_{d_i+1}} - e^{x_k}} \right) q^{d_j - d_{j-1}} \\ &= \sigma(\mathbf{d}, \mathbf{d}_i^+) \left( p \text{ch} \prod_{k \in I_i} \frac{q X_{d_i+1} - q^{-1} X_k}{X_{d_i+1} - X_k} \right) \\ &= \text{ch}(E_{i,0} \pi) \end{aligned}$$

The proof for the rest of the generators is identical.  $\square$

**7.20. Proof of Lemma 7.17.** Let  $\Delta(u)$  be the formal series given in Lemma 7.16 and set

$$\mathfrak{z}(u) = \hbar \sum_{r \geq 0} \mathfrak{z}_r u^{-r-1} = \log(\Delta(u))$$

For any  $j \in \mathbf{J}$ , define  $y_j(u) \in Y^0[[u^{-1}]]$  by

$$y_j(u) = \mathfrak{z}(u + (j-1)\hbar) + d_j(u) + \sum_{s=1}^{j-1} (d_{j-s}(u + s\hbar) - d_{j-s}(u + (s-1)\hbar)) \quad (7.3)$$

A computation similar to the one given in 7.19 shows that for any  $j \in \mathbf{J}$ ,

$$\Psi_Y(B(y_j(u))) = \frac{1 - e^{\hbar v}}{v} \sum_{k \in I_j} e^{x_k v} \quad (7.4)$$

Set now<sup>2</sup>

$$J(v) = \log \left( \frac{v}{1 - e^{-v}} \right) \in \mathbb{Q}[[v]]$$

and, for any  $i \in \mathbf{I}$ , define

$$\mathrm{td}_i^+(v) = By_i(-\partial)J'(v + \hbar) \quad (7.5)$$

$$\mathrm{td}_i^-(v) = -By_{i+1}(-\partial)J'(v) \quad (7.6)$$

where  $\partial = d/dv$ . We claim that  $\mathrm{Td}_i^\pm(u) = \exp(\mathrm{td}_i^\pm(u))$  satisfy the conditions of the Lemma. By (7.4) we have

$$\Psi_Y(\mathrm{td}_i^+(v)) = \left( \frac{1 - e^{-\hbar \partial}}{-\partial} \sum_{k \in I_i} e^{-x_k \partial} \right) \partial J(v + \hbar)$$

Using formal Taylor expansions, *i.e.*,  $e^{-p\partial}f(v) = f(v - p)$ , we get

$$\begin{aligned} \Psi_Y(\mathrm{td}_i^+(v)) &= \sum_{k \in I_i} J(v - x_k) - J(v - x_k + \hbar) \\ &= \sum_{k \in I_i} \log \left( \frac{v - x_k}{1 - e^{-v+x_k}} \frac{1 - e^{-v+x_k-\hbar}}{v - x_k + \hbar} \right) \\ &= \log \left( \prod_{k \in I_i} \frac{v - x_k}{1 - e^{-v+x_k}} \frac{1 - e^{-v+x_k+\hbar}}{v - x_k + \hbar} \right) \end{aligned}$$

The proof for the  $-$  case is same.

**7.21. Standard form of  $\Phi$ .** We rewrite below the assignment  $\Phi$  in the form in which Theorem 7.6 can be applied, and use this to prove that  $\Phi$  extends to an algebra homomorphism  $U_\hbar(L\mathfrak{gl}_n) \rightarrow \widehat{Y_\hbar(\mathfrak{gl}_n)}$ .

**Lemma.** *Let  $y_j(v)$  be given by (7.3), and  $\lambda_i^\pm(u)$  the operators of Proposition 7.5. Then,*

$$(\lambda_i^\pm(u) - 1)By_j(v) = \pm(\delta_{i,j} - \delta_{j,i+1}) \frac{e^{\hbar v} - 1}{v} e^{uv}$$

The proof of this lemma essentially follows from Proposition 7.5.

---

<sup>2</sup>Note the difference between  $J(v)$  and the function  $G(v)$  used in Section 4.4 for constructing the solutions for simple Lie algebras:  $J(v) = G(v) + \frac{v}{2}$ .

**Corollary.** *For any  $i, i' \in \mathbf{I}$ , we have*

$$\begin{aligned} \frac{\lambda_i^+(u)(\text{Td}_{i'}^+(v))}{\text{Td}_{i'}^+(u)} &= \frac{\text{Td}_{i'}^+(v)}{\lambda_i^-(u)(\text{Td}_{i'}^+(v))} = \left( \frac{v-u+\hbar}{1-e^{-v+u-\hbar}} \frac{1-e^{-v+u}}{v-u} \right)^{\delta_{i,i'}-\delta_{i,i'-1}} \\ \frac{\lambda_i^+(u)(\text{Td}_{i'}^-(v))}{\text{Td}_{i'}^-(u)} &= \frac{\text{Td}_{i'}^-(v)}{\lambda_i^-(u)(\text{Td}_{i'}^-(v))} = \left( \frac{v-u}{1-e^{-v+u}} \frac{1-e^{-v+u+\hbar}}{v-u-\hbar} \right)^{\delta_{i,i'}-\delta_{i,i'+1}} \end{aligned}$$

It follows that

$$\begin{aligned} \Phi(E_{i,k}) &= e^{k\sigma_i^+} g_i^+(\sigma_i^+) e_{i,0} \\ \Phi(F_{i,k}) &= e^{k\sigma_i^-} g_i^-(\sigma_i^-) f_{i,0} \end{aligned}$$

where

$$\begin{aligned} g_i^+(u) &= q^{-\Delta_0-\theta_{i,0}} \frac{\hbar}{q-q^{-1}} \text{Td}_i^+(u) \\ g_i^-(u) &= q^{\Delta_0+\theta_{i+1,0}} \frac{\hbar}{q-q^{-1}} \text{Td}_i^-(u) \end{aligned} \tag{7.7}$$

7.22. We record the action of the operators  $\lambda_{i'}^\pm(u)$  on  $g_i^\pm(v)$  using Corollary 7.21

$$\lambda_i^+(u)(g_i^+(v)) = \lambda_{i-1}^-(u)(g_i^+(v)) = g_i^+(v) \frac{v-u+\hbar}{e^{v+\hbar/2}-e^{u-\hbar/2}} \frac{e^v-e^u}{v-u} \tag{7.8}$$

$$\lambda_{i+1}^+(u)(g_i^-(v)) = \lambda_i^-(u)(g_i^-(v)) = g_i^-(v) \frac{v-u-\hbar}{e^{v-\hbar/2}-e^{u+\hbar/2}} \frac{e^v-e^u}{v-u} \tag{7.9}$$

Using the fact that  $\lambda_i^+(u)\lambda_i^-(u) = \text{Id}$ , we get four more equations from these. Moreover,  $\lambda_{i'}^\pm(u)(g_i^\pm(v)) = g_i^\pm(v)$  for  $i' \neq i, i-1$  and  $\lambda_{i'}^\pm(u)(g_i^\pm(v)) = g_i^\pm(v)$  for  $i' \neq i, i+1$ .

7.23.

**Theorem.** *The series  $g_i^\pm(u)$  satisfy the conditions (A), (B), (C0)–(C2) of Theorem 7.6 and therefore give rise to an algebra homomorphism  $\Phi : U_\hbar(L\mathfrak{gl}_n) \rightarrow \widehat{Y_\hbar(\mathfrak{gl}_n)}$ .*

7.24. **Proof of (A).** We need to prove that for every  $i, i' \in \mathbf{I}$ , we have

$$g_i^+(u)\lambda_i^+(u)(g_{i'}^-(v)) = g_{i'}^-(v)\lambda_{i'}^-(v)(g_i^+(u))$$

If  $i \neq i', i'+1$ , both sides are equal to  $g_i^+(u)g_{i'}^-(v)$ . For  $i = i'$ , by (7.8)–(7.9), the left and right-hand sides are respectively equal to

$$\frac{v-u}{e^v-e^u} \frac{e^{v-\hbar/2}-e^{u+\hbar/2}}{v-u-\hbar} \quad \text{and} \quad \frac{u-v}{e^u-e^v} \frac{e^{u+\hbar/2}-e^{v-\hbar/2}}{u-v+\hbar}$$

The case  $i = i' + 1$  follows in the same way.

**7.25. Proof of (B).** Let  $i \in \mathbf{I}$ . By (7.9),

$$\begin{aligned} g_i^+(u) \lambda_i^+(u) (g_i^-(u)) &= g_i^+(u) g_i^-(u) \frac{q - q^{-1}}{\hbar} \\ &= q^{\theta_{i+1,0} - \theta_{i,0}} \frac{\hbar}{q - q^{-1}} \text{Td}_i^+(u) \text{Td}_i^-(u) \\ &= \frac{\hbar}{q - q^{-1}} \exp \left( \frac{\hbar(\theta_{i+1,0} - \theta_{i,0})}{2} + td_i^+(u) + td_i^-(u) \right) \end{aligned}$$

By definition of  $td_i^\pm$ ,

$$\begin{aligned} td_i^+(u) + td_i^-(u) &= -By_{i+1}(-\partial)J'(u) + By_i(-\partial)J'(u + \hbar) \\ &= -B(y_{i+1}(v) - y_i(v + \hbar))|_{v=-\partial} J'(u) \end{aligned}$$

where the second equality follows from  $J'(u + \hbar) = e^{\hbar\partial} J'(u)$  and the fact that  $e^{pv} B(f(u)) = B(f(u + p))$ . Next, the definition of  $y_i$  yields

$$y_{i+1}(v) - y_i(v + \hbar) = d_{i+1}(v) - d_i(v)$$

hence

$$td_i^+(u) + td_i^-(u) = \hbar \sum_{r \geq 0} (-1)^{r+1} \frac{d_{i+1,r} - d_{i,r}}{r!} J^{(r+1)}(u)$$

which implies that

$$\frac{\hbar(\theta_{i+1,0} - \theta_{i,0})}{2} + td_i^+(u) + td_i^-(u) = \hbar \sum_{r \geq 0} (-1)^{r+1} \frac{d_{i+1,r} - d_{i,r}}{r!} G^{(r+1)}(u)$$

and the proof of (B) follows from Proposition 4.5

**7.26. Proof of (C0)–(C2).** The condition (C0) follows from the fact that  $(\lambda_i^\pm(u) - 1)(g_{i'}^\pm(v)) = 0$  if  $|i - i'| > 1$ . Since the proof of (C1) is the same as the one given in the verification of (A), we are left with checking (C2). We need to show that, for any  $i \in \mathbf{I} \setminus \{n - 1\}$ ,

$$g_i^+(u) \lambda_i^+(u) g_{i+1}^+(v) \frac{e^u - e^v}{u - v} = g_{i+1}^+(v) \lambda_{i+1}^\pm(v) g_i^+(u) \frac{e^{u-\hbar/2} - e^{v+\hbar/2}}{u - v - \hbar}$$

Using (7.8)–(7.9), the left and right-hand sides are respectively equal to

$$\frac{u - v}{e^u - e^v} \frac{e^{u-\hbar/2} - e^{v+\hbar/2}}{u - v - \hbar} \frac{e^u - e^v}{u - v} \quad \text{and} \quad \frac{e^{u-\hbar/2} - e^{v+\hbar/2}}{u - v - \hbar}$$

## APPENDIX A. PROOF OF THE SERRE RELATIONS

**A.1.** Let  $\mathfrak{g}$  be a complex, semisimple Lie algebra. The aim of this appendix is to prove the following

**Proposition.** *Let  $\Phi$  be the assignment  $\{E_{i,k}, F_{i,k}, H_{i,k}\} \rightarrow \widehat{Y_\hbar(\mathfrak{g})}$  given in Sections 3.1–3.2, and assume that the relations (A) and (B) of Theorem 3.3 hold. Then,  $\Phi$  is compatible with the  $q$ -Serre relations (QL6).*

For  $i \neq j \in \mathbf{I}$ , set  $m = 1 - a_{ij}$ . Define, for any  $\underline{k} = (k_1, \dots, k_m) \in \mathbb{Z}^m$  and  $l \in \mathbb{Z}$

$$\mathcal{S}_{ij}^q(\underline{k}, l) = \sum_{\pi \in \mathfrak{S}_m} \sum_{s=0}^m (-1)^s \begin{bmatrix} m \\ s \end{bmatrix}_{q_i} \Phi(E_{i, k_{\pi(1)}}) \cdots \Phi(E_{i, k_{\pi(m-s)}}) \Phi(E_{j, l}) \Phi(E_{i, k_{\pi(m-s+1)}}) \cdots \Phi(E_{i, k_{\pi(m)}}) \in \widehat{Y_h(\mathfrak{g})} \quad (\text{A.1})$$

and let  $\mathcal{S}_{ij}^q = \mathcal{S}_{ij}^q(\underline{0}, 0)$ , explicitly given as follows

$$\mathcal{S}_{ij}^q = \sum_{s=0}^m (-1)^s \begin{bmatrix} m \\ s \end{bmatrix}_{q_i} (\Phi(E_{i,0}))^{m-s} \Phi(E_{j,0}) (\Phi(E_{i,0}))^s \quad (\text{A.2})$$

Our aim is to show that  $\mathcal{S}_{ij}^q(\underline{k}, l) = 0$ . Let us outline the the main steps of the proof.

- (1) We first reduce the proof of  $\mathcal{S}_{ij}^q(\underline{k}, l) = 0$  to  $\mathcal{S}_{ij}^q = 0$ . This is achieved in Lemma A.5.
- (2) By a standard argument using the representation theory of  $U_h \mathfrak{sl}_2$ , we deduce in Lemma A.6 that  $\mathcal{S}_{ij}^q$  acts by zero on any finite-dimensional representation of  $Y_h(\mathfrak{g})$ .
- (3) Finally, we show that these representations separate points in  $Y_h(\mathfrak{g})$ , and hence that  $\mathcal{S}_{ij}^q = 0$ . §A.8 and A.9 are devoted to the proof of this fact (Corollary A.9) which was communicated to us by V. G. Drinfeld.

**A.2. The algebra  $\overline{Y}$ .** Define an auxiliary algebra  $\overline{Y}$  to be the unital, associative  $\mathbb{C}[\hbar]$ -algebra generated by  $\{\overline{\xi}_{i,r}, \overline{x}_{i,r}\}_{i \in \mathbf{I}, r \in \mathbb{N}}$  subject to the following relations

- (1) For every  $i, j \in \mathbf{I}$  and  $r, s \in \mathbb{N}$

$$[\overline{\xi}_{i,r}, \overline{\xi}_{j,s}] = 0$$

- (2) For every  $i, j \in \mathbf{I}$  and  $s \in \mathbb{N}$

$$[\overline{\xi}_{i,0}, \overline{x}_{j,s}] = d_i a_{ij} \overline{x}_{j,s}$$

- (3) For every  $i, j \in \mathbf{I}$  and  $r, s \in \mathbf{I}$

$$[\overline{\xi}_{i,r+1}, \overline{x}_{j,s}] - [\overline{\xi}_{i,r}, \overline{x}_{j,s+1}] = \frac{d_i a_{ij} \hbar}{2} (\overline{\xi}_{i,r} \overline{x}_{j,s} + \overline{x}_{j,s} \overline{\xi}_{i,r})$$

We denote by  $\overline{Y}^0 \subset \overline{Y}$  the commutative subalgebra generated by  $\{\overline{\xi}_{i,r}\}_{i \in \mathbf{I}, r \in \mathbb{N}}$  and by  $\overline{Y}^{>0}$  the subalgebra of  $\overline{Y}$  generated by  $\{\overline{x}_{i,r}\}_{i \in \mathbf{I}, r \in \mathbb{N}}$ . The latter is a free  $\mathbb{C}[\hbar]$ -algebra over this set of generators. Moreover, by Corollary 2.5,  $\overline{Y} \cong \overline{Y}^0 \otimes \overline{Y}^{>0}$ .

**A.3. The operators  $\overline{\sigma}_{i,(k)}$  and  $\overline{\sigma}_j$ .** The algebra  $\overline{Y}$  has a grading by the root lattice  $Q$  given by

$$\deg(\overline{\xi}_{i,r}) = 0 \quad \text{and} \quad \deg(\overline{x}_{i,r}) = \alpha_i$$

Fix henceforth  $i \neq j \in \mathbf{I}$ , set  $m = 1 - a_{ij}$  and let  $\overline{Y}_{m\alpha_i + \alpha_j}$  be the homogeneous component of  $\overline{Y}$  of degree  $m\alpha_i + \alpha_j$ .

Define operators  $\bar{\sigma}_j, \bar{\sigma}_{i,(k)}$  on  $\bar{Y}_{m\alpha_i+\alpha_j}$  as follows. Since  $\bar{Y}_{m\alpha_i+\alpha_j} \cong \bar{Y}^0 \otimes \bar{Y}_{m\alpha_i+\alpha_j}^{>0}$  and  $\bar{Y}^{>0}$  is free, we have

$$\bar{Y}_{m\alpha_i+\alpha_j}^{>0} \cong \bar{Y}^0 \otimes \bigoplus_{s=0}^m \bar{Y}(i)^{\otimes m-s} \otimes \bar{Y}(j) \otimes \bar{Y}(i)^{\otimes s}$$

where, for  $a = i, j$ ,  $\bar{Y}(a) = \bar{Y}_{\alpha_a}^{>0}$  is spanned by  $\{\bar{x}_{a,r}\}_{r \in \mathbb{N}}$ . Let  $\bar{\sigma}_a$  denote the  $\mathbb{C}[\hbar]$ -linear map on  $\bar{Y}(a)$  given by  $\bar{\sigma}_a(x_{a,r}) = x_{a,r+1}$ . For any  $k = 1, \dots, m$ , define the  $Y^0$ -linear operator  $\bar{\sigma}_{i,(k)}$  on  $\bar{Y}_{m\alpha_i+\alpha_j}$  by letting it act on the summand  $\bar{Y}(i)^{\otimes m-s} \otimes \bar{Y}(j) \otimes \bar{Y}^{\otimes s}(i)$  as

$$\bar{\sigma}_{i,(k)} = \begin{cases} 1^{\otimes k-1} \otimes \bar{\sigma}_i \otimes 1^{\otimes m+1-k} & \text{if } k \leq m-s \\ 1^{\otimes k} \otimes \bar{\sigma}_i \otimes 1^{\otimes m-k} & \text{otherwise} \end{cases}$$

Similarly, let  $\sigma_j \in \text{End}_{\bar{Y}^0}(\bar{Y}_{m\alpha_i+\alpha_j})$  be given by  $1^{\otimes m-s} \otimes \bar{\sigma}_j \otimes 1^{\otimes s}$  on  $\bar{Y}(i)^{\otimes m-s} \otimes \bar{Y}(j) \otimes \bar{Y}^{\otimes s}(i)$ .

**A.4. The projection  $p_{ij}$ .** Let  $p : \bar{Y} \rightarrow Y_h(\mathfrak{g})$  be the algebra homomorphism obtained by sending  $\bar{\xi}_{a,r} \mapsto \xi_{a,r}$  and  $\bar{x}_{a,r} \mapsto x_{a,r}^+$  for every  $a \in \mathbf{I}$  and  $r \in \mathbb{N}$ , and let  $p_{ij}$  be the restriction of  $p$  to  $\bar{Y}_{m\alpha_i+\alpha_j}$ . The following holds by Proposition 2.8.

**Lemma.** *The kernel of  $p_{ij}$  is the  $\mathbb{C}[\hbar]$ -linear span of the following elements*

$$(1) \text{ For any } 0 \leq s \leq m-1 \text{ and } A(u_1, \dots, u_m, w) \in \bar{Y}^0[u_1, \dots, u_m, w]$$

$$A(\bar{\sigma}_{i,(1)}, \dots, \bar{\sigma}_{i,(m)}, \bar{\sigma}_j) \left( (\bar{\sigma}_{i,(m-s)} - \bar{\sigma}_j - a\hbar) \bar{x}_{i,0}^{m-s} \bar{x}_{j,0} \bar{x}_{i,0}^s \right. \\ \left. - (\bar{\sigma}_{i,(m-s)} - \bar{\sigma}_j + a\hbar) \bar{x}_{i,0}^{m-s-1} \bar{x}_{j,0} \bar{x}_{i,0}^{s+1} \right)$$

where  $a = d_i a_{ij}/2$ .

$$(2) \text{ For any } 0 \leq s \leq m, k \in \{1, \dots, m-1\} \setminus \{m-s\} \text{ and } A(u_1, \dots, u_m, w) \in \bar{Y}^0[u_1, \dots, u_m, w]^{(k, k+1)}$$

$$A(\bar{\sigma}_{i,(1)}, \dots, \bar{\sigma}_{i,(m)}, \bar{\sigma}_j) (\bar{\sigma}_{i,(k)} - \bar{\sigma}_{i,(k+1)} - d_i \hbar) \bar{x}_{i,0}^{m-s} \bar{x}_{j,0} \bar{x}_{i,0}^s$$

$$(3) \text{ For every } A(u_1, \dots, u_m, w) \in \bar{Y}^0[u_1, \dots, u_m, w]^{\mathfrak{S}_m}$$

$$A(\bar{\sigma}_{i,(1)}, \dots, \bar{\sigma}_{i,(m)}, \bar{\sigma}_j) \left( \sum_{s=0}^m (-1)^s \binom{m}{s} \bar{x}_{i,0}^{m-s} \bar{x}_{j,0} \bar{x}_{i,0}^s \right)$$

**Corollary.** *Let  $X \in \text{Ker}(p_{ij})$  and  $A(u_1, \dots, u_m, w) \in \bar{Y}^0[u_1, \dots, u_m, w]^{\mathfrak{S}_m}$ . Then,*

$$A(\bar{\sigma}_{i,(1)}, \dots, \bar{\sigma}_{i,(m)}, \bar{\sigma}_j) X \in \text{Ker}(p_{ij})$$



**A.5. Reduction step.** Let  $\overline{\mathcal{S}}_{ij}^q(\underline{k}, l), \overline{\mathcal{S}}_{ij}^q$  denote the elements of  $\overline{Y}_{m\alpha_i + \alpha_j}$  defined by the same expressions as (A.1)–(A.2). Then,

$$\overline{\mathcal{S}}_{ij}^q(\underline{k}, l) = \left( \sum_{\pi \in \mathfrak{S}_m} e^{k_{\pi(1)} \overline{\sigma}_{i,(1)}} \dots e^{k_{\pi(m)} \overline{\sigma}_{i,(m)}} e^{l \overline{\sigma}_j} \right) \overline{\mathcal{S}}_{ij}^q$$

Using Corollary A.4, we obtain the following

**Lemma.**  $\mathcal{S}_{ij}^q = 0$  implies  $\mathcal{S}_{ij}^q(\underline{k}, l) = 0$  for every  $k_1, \dots, k_m, l \in \mathbb{Z}$ .

**A.6.** By a finite-dimensional representation of  $\widehat{Y_h(\mathfrak{g})}$  we shall mean a finitely-generated topologically free  $\mathbb{C}[[\hbar]]$ -module endowed with a  $\mathbb{C}[[\hbar]]$ -linear action of  $\widehat{Y_h(\mathfrak{g})}$ .

**Lemma.** Let  $\mathcal{V}$  be a finite-dimensional representation of  $\widehat{Y_h(\mathfrak{g})}$ . Then,  $\mathcal{S}_{ij}^q$  acts by zero on  $\mathcal{V}$ .

PROOF. Let  $\mathcal{U}_i$  be the subalgebra of  $\widehat{Y_h(\mathfrak{g})}$  generated by

$$\mathcal{E}_i = \Phi(E_{i,0}) \quad \mathcal{F}_i = \Phi(F_{i,0}) \quad \mathcal{H}_i = \Phi(H_{i,0})$$

By Lemma 3.4,  $\{\mathcal{E}_i, \mathcal{F}_i, \mathcal{H}_i\}$  satisfy the defining relations of the quantum group  $U_{\hbar_i} \mathfrak{sl}_2$ , where  $\hbar_i = d_i \hbar / 2$ . We use the following notation of  $q$ -adjoint operator (see [12, §4.18]) which gives a representation of  $\mathcal{U}_i$  on any algebra containing it

$$\begin{aligned} \text{ad}_q(\mathcal{E}_i)(X) &= \mathcal{E}_i X - \mathcal{K}_i X \mathcal{K}_i^{-1} \mathcal{E}_i \\ \text{ad}_q(\mathcal{F}_i)(X) &= \mathcal{F}_i X \mathcal{K}_i - X \mathcal{F}_i \mathcal{K}_i \\ \text{ad}_q(\mathcal{H}_i)(X) &= [\mathcal{H}_i, X] \end{aligned}$$

where  $\mathcal{K}_i = q_i^{\mathcal{H}_i} = e^{\hbar_i \mathcal{H}_i}$ . Let  $\rho : \widehat{Y_h(\mathfrak{g})} \rightarrow \text{End}(\mathcal{V})$  be the representation. Then,

$$\begin{aligned} \text{ad}_q(\rho(\mathcal{F}_i))\rho(\mathcal{E}_j) &= 0 \\ \text{ad}_q(\rho(\mathcal{H}_i))\rho(\mathcal{E}_j) &= a_{ij}\rho(\mathcal{E}_j) \end{aligned}$$

where the first identity follows from Lemma 3.4. Thus, as a  $\mathcal{U}_i$ -module,  $\text{End}(\mathcal{V})$  contains the lowest weight vector  $\rho(\mathcal{E}_j)$  of weight  $a_{ij}$ . By the representation theory of  $U_{\hbar_i} \mathfrak{sl}_2$ , we get

$$\text{ad}_q(\rho(\mathcal{E}_i))^m \rho(\mathcal{E}_j) = \rho(\text{ad}_q(\mathcal{E}_i)^m \mathcal{E}_j) = 0$$

and the assertion follows from the well-known identity (see [12, Lemma 4.18])

$$\text{ad}_q(\mathcal{E}_i)^m \mathcal{E}_j = \sum_{s=0}^m (-1)^s \begin{bmatrix} m \\ s \end{bmatrix}_{q_i} \mathcal{E}_i^{m-s} \mathcal{E}_j \mathcal{E}_i^s$$

□

A.7. Let  $I_{\hbar} \subset Y_{\hbar}(\mathfrak{g})$  be the ideal defined by

$$I_{\hbar} = \bigcap_{(\mathcal{V}, \rho)} \text{Ker}(\rho)$$

where  $\mathcal{V}$  runs over all finite-dimensional *graded* modules over  $Y_{\hbar}(\mathfrak{g})$ , that is finitely-generated torsion-free  $\mathbb{C}[\hbar]$ -modules admitting a  $\mathbb{C}[\hbar]$ -linear action  $\rho : Y_{\hbar}(\mathfrak{g}) \rightarrow \text{End}(\mathcal{V})$  and a  $\mathbb{Z}$ -grading compatible with that on  $Y_{\hbar}(\mathfrak{g})$ .

**Lemma.**  $I_{\hbar} = 0$  implies  $\mathcal{S}_{ij}^q = 0$ .

PROOF. The action of  $Y_{\hbar}(\mathfrak{g})$  on any finite-dimensional graded module  $\mathcal{V}$  extends to one of  $\widehat{Y_{\hbar}(\mathfrak{g})}$  on the completion  $\widehat{\mathcal{V}}$  of  $\mathcal{V}$  with respect to its grading. By Lemma A.6,  $\mathcal{S}_{ij}^q$  acts by 0 on  $\widehat{\mathcal{V}}$  and therefore so do its homogeneous components  $\mathcal{S}_{ij;n}^q \in Y_{\hbar}(\mathfrak{g})$ ,  $n \geq 0$  on  $\mathcal{V}$ . Thus,  $\mathcal{S}_{ij;n}^q \in I_{\hbar}$  for any  $n$  and  $\mathcal{S}_{ij}^q = 0$ .  $\square$

A.8. The following result, and its proof are due to V. Drinfeld [8]

**Proposition.** *The ideal  $I_{\hbar} \subset Y_{\hbar}(\mathfrak{g})$  is trivial.*

PROOF. It suffices to show that  $I = I_{\hbar}/\hbar I_{\hbar}$  is trivial. Indeed, if  $I_{\hbar} = \hbar I_{\hbar}$ , then  $I_{\hbar} = \bigcap_k \hbar^k I_{\hbar} \subset \bigcap_k \hbar^k Y_{\hbar}(\mathfrak{g}) = 0$ . By definition of  $I_{\hbar}$ ,  $I_{\hbar} \cap \hbar Y_{\hbar}(\mathfrak{g}) = \hbar Y_{\hbar}(\mathfrak{g})$  so that  $I$  embeds into  $Y_{\hbar}(\mathfrak{g})/\hbar Y_{\hbar}(\mathfrak{g}) = U(\mathfrak{g}[s])$ . Since graded representations are stable under tensor product,  $I_{\hbar}$  is a Hopf ideal of  $Y_{\hbar}(\mathfrak{g})$ , that is

$$\Delta(I_{\hbar}) \subset Y_{\hbar}(\mathfrak{g}) \otimes I_{\hbar} + I_{\hbar} \otimes Y_{\hbar}(\mathfrak{g})$$

It follows that  $I$  is a co-Poisson Hopf ideal of  $U(\mathfrak{g}[s])$ . By Corollary A.9 below, any such ideal is either trivial or equal to  $U(\mathfrak{g}[s])$ . Since  $Y_{\hbar}(\mathfrak{g})$  possesses non-trivial finite-dimensional graded representations, for example the action on  $\mathbb{C}[\hbar]$  given by the counit,  $I_{\hbar}$  is a proper ideal of  $Y_{\hbar}(\mathfrak{g})$  and is therefore equal to zero.  $\square$

A.9. Recall that a co-Poisson Hopf algebra  $A$  is a Hopf algebra together with a Poisson cobracket  $\delta : A \rightarrow A \wedge A$  satisfying the following compatibility condition (see [3, §6.2] for details):

$$\delta(xy) = \delta(x)\Delta(y) + \Delta(x)\delta(y)$$

For a Lie algebra  $\mathfrak{a}$ , there is a one to one correspondence between co-Poisson structures on  $U\mathfrak{a}$  and Lie bialgebra structures on  $\mathfrak{a}$  [3, Proposition 6.2.3]. Moreover, there is a one to one correspondence between co-Poisson Hopf ideals of  $U\mathfrak{a}$  and Lie bialgebra ideals of  $\mathfrak{a}$ .

The Lie bialgebra structure on  $\mathfrak{g}[s]$  is given by

$$\begin{aligned} \delta : \mathfrak{g}[s] &\rightarrow \mathfrak{g}[s] \otimes \mathfrak{g}[s] \cong (\mathfrak{g} \otimes \mathfrak{g})[s, t] \\ \delta(f)(s, t) &= (\text{ad}(f(s)) \otimes 1 + 1 \otimes \text{ad}(f(t))) \left( \frac{\Omega}{s - t} \right) \end{aligned} \quad (\text{A.3})$$

where  $\Omega \in \mathfrak{g} \otimes \mathfrak{g}$  is the Casimir tensor. Note that  $\delta$  lowers the degree by 1.

Let  $\mathfrak{a} \subset \mathfrak{g}[s]$  be the Lie bialgebra ideal corresponding to co-Poisson Hopf ideal  $I \subset U(\mathfrak{g}[s])$ . By the discussion given in previous paragraph,

$$\delta(\mathfrak{a}) \subset \mathfrak{a} \otimes \mathfrak{g}[s] + \mathfrak{g}[s] \otimes \mathfrak{a} \quad (\text{A.4})$$

**Lemma.** *Let  $\mathfrak{a} \subset \mathfrak{g}[s]$  be an ideal. Then  $\mathfrak{a}$  is of the form  $\mathfrak{a} = \mathfrak{g} \otimes g\mathbb{C}[s]$  for some polynomial  $g \in \mathbb{C}[s]$ .*

PROOF. Let  $S \subset \mathbb{C}[s]$  be the set of all polynomials  $f$  such that there exists some non zero  $x \in \mathfrak{g}$  for which  $x \otimes f \in \mathfrak{a}$ . We claim that  $S$  is an ideal of  $\mathbb{C}[s]$ . Let  $f \in S$  and  $g \in \mathbb{C}[s]$ . Let  $0 \neq x \in \mathfrak{g}$  be such that  $x \otimes f \in \mathfrak{a}$ , and choose  $y \in \mathfrak{g}$  such that  $[x, y] \neq 0$ . Then

$$[x, y] \otimes fg = [x \otimes f, y \otimes g] \in \mathfrak{a}$$

and hence  $fg \in S$ .

Now for any  $f \in S$ , the set  $\{x \in \mathfrak{g} : x \otimes f \in \mathfrak{a}\}$  is an ideal in  $\mathfrak{g}$ , which is non-zero and hence equal to  $\mathfrak{g}$ . This proves that  $\mathfrak{a} = \mathfrak{g} \otimes S$ . Since  $\mathbb{C}[s]$  is a principal ideal domain, the lemma is proved.  $\square$

**Corollary.** *Let  $\mathfrak{a} \subset \mathfrak{g}[s]$  be a Lie bialgebra ideal. Then either  $\mathfrak{a} = 0$  or  $\mathfrak{a} = \mathfrak{g}[s]$ .*

PROOF. Let  $g \in \mathbb{C}[s]$  be such that  $\mathfrak{a} = \mathfrak{g} \otimes (g) \subset \mathfrak{g}[s]$ . By (A.3), we know that the Lie cobracket  $\delta$  lowers the degree by 1. Using (A.4), we conclude that  $g$  is a constant polynomial.  $\square$

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